DIFFUSION INTEREST RATE MODELS IN ACTUARIAL COMPUTATIONS

Jarolímková Tereza

Organisations:
Faculty of Mathematics and Physics, Charles University Prague,
Česká pojišťovna, a.s.,

Address: Spálená 16, 110 000 Prague 1, Czech republic
Telephone: +420 224 053 228
Fax: +420 224 053 267
E-mail: tjarolimkova@cpoj.cz

Abstract:
The paper describes how to use diffusion interest rate models in practical actuarial computations. It concentrates on numerical methods exploiting the Markovian character of diffusion processes. We do not use the assumption that investment yields in particular accounting periods are non-correlated, which is common in approximative methods that can be found in literature. The method is applied to valuation of the liability and liability adequacy testing for an endowment contract with profit sharing.

Keywords:
Diffusion process, Markovian character, Interest rate model, Valuation of life insurance liability.

1 INTRODUCTION

The stochastic models become an important tool in a practical actuarial modelling, focusing especially on guarantees pricing and liability valuation. The aim of the paper is to propose a method based on the Markovian character of diffusion processes, which can be used in practical actuarial computations. We will apply the method to endowment contract with profit sharing, concerning on liability valuation. The method will be illustrated by numerical examples which were calculated by a model created in MS Excel.

1.1 Risk approach

There are two main components of risk which are important for our approach. These are volatility and uncertainty. The volatility risk can be diversified, in case risks are independent. A choice of a model and setting parameters constitute the uncertainty risk.

The main principles used for actuarial modelling are recommended by International Association of Actuaries (IAA) in [7]. The insurer is exposed especially to the technical and the investment risk. We will assume that the technical risk can be diversified. It is proper to use a stochastic model to allow for the volatility of the investment risk. International Association of Actuaries also declares the principle of parsimony, which consist of using simple models. These models can by easily interpreted and used for practical calculations.

1.2 Fair value principles

Our model will be based on the fair value concept, which is the part of IASB a FASB framework. The key elements are the time value of money and allowing for market value margins. We will respect the three main assumptions of the fair value concept

- Risk free rate of interest is used for discounting cash flows.
- The expected cash flows should be adjusted by market value margins in order to include the market value of risk connected with an adverse deviation of assumptions (referred also as an adjustment for risk and uncertainty).
- The adjustment should include imperfections of the market. This approach can be found in Towers Perrin’s study referred as [5] .

2 INSTANTANEOUS RATE OF INTEREST

We will suppose the space $\Omega = C^1_{[0,T]} \times C^2_{[0,T]}$ with probability measure $w^1 \times w^2$ representing trajectories of two Wiener processes $\{W_t^1, W_t^2\}$.

For modelling the instantaneous rate of interest we will use a diffusion process $\{r_t\}$, which can be expressed by the equation $dr_t = \theta(t, r_t)dt + \delta(t, r_t)dW_t^1$, $t \in [0,T]$, where θ and δ are generally functions and W_t^1 is Wiener process regarding to a filtration $F = \{F_t, t \in [0,T]\}$.

The Markovian character of diffusion processes is very important for our valuation method. It says that on condition that $r_s = y$ are $\{r_t, t \geq s\}$ and $\{r_u, u \leq s\}$ independent, as is shown in [4].

2.1 Risk free rate of interest

Denote I_k a random variable representing the risk free rate of interest in the k^{th} policy year. We assume that

$$(1)\ \ 1 + I_k = e^{\int_{k-1}^{k} r_t\, dt} = e^{(R_k - R_{k-1})}$$

and that $1 + I_k \sim \log N(\mu_k, \sigma_k{}^2)$, where parameters μ_k a σ_k^2 depend on a model used for $\{r_t\}$ and will be determined later.

We will use the Hull-White model

$$(2)\ \ dr_t = (\rho(t) - \kappa \cdot r_t)dt + \delta_1\ dW_t^1,\ t \in [0,T],$$

for modelling the instantaneous risk free rate of interest r_t.

We will use $v^{(k)} = e^{-\int_0^k r_u du}$ for discounting cash flows from the end of the k^{th} policy year to the time of the policy issue.

2.2 Parameters of the model

We will determine parameters of the model in compliance with the value of zero-coupon bond with maturity T expressed as (see [1])

$$(3)\ \ P(r,t,T) = E\left\{ e^{-\int_t^T r_s\, ds} \,\middle|\, r_t = r \right\} .$$

Let $f^M(0,t)$ denote a market yield curve. Assuming $f^M(0,t) = -\frac{\partial}{\partial t} \ln P(r,0,t)$, we can express the parameter $\rho(t)$ given by the equation (2) as

$$\rho(t) = f^M(0,t)' + \kappa \cdot f^M(0,t) + \frac{\delta_1^2}{2\cdot\kappa^2}\cdot\left(1 - e^{-2\cdot\kappa\cdot t}\right) \text{ and } r_0 = f^M(0,0) .$$

It is useful to consider a simple function $f^M(0,t)$ for further analytical calculation, for example $f^M(0,t) = \alpha + \beta\cdot\left(1 - e^{-\frac{t}{\gamma}}\right)$.

Thus $\rho(t) = \left[\kappa\cdot(\alpha+\beta) + \frac{\delta_1^2}{2\cdot\kappa}\right] + \left(\frac{\beta}{\gamma} - \kappa\cdot\beta\right)\cdot e^{-\frac{t}{\gamma}} - \frac{\delta_1^2}{2\cdot\kappa}\cdot e^{-2\cdot\kappa\cdot t}$.

2.3 Characteristics of r_k a R_k

Solving the differential equation (2), the instantaneous rate of interest r_k can be expressed as

$$(4)\ \ r_k = e^{-\kappa\cdot k}\cdot r_0 + \int_0^k e^{-\kappa\cdot(k-u)}\rho(u)du + \delta_1\cdot\int_0^k e^{-\kappa\cdot(k-u)}dW_u^1 ,\ r_0 = s .$$

The variable R_k equals to

$$R_k = \int_0^k r_u du = \frac{1}{\kappa}\cdot\left(1 - e^{-\kappa\cdot k}\right)\cdot r_0 + \frac{1}{\kappa}\cdot\int_0^k \left(1 - e^{-\kappa\cdot(k-u)}\right)\rho(u)du + \frac{\delta_1}{\kappa}\cdot\int_0^k \left(1 - e^{-\kappa\cdot(k-u)}\right)dW_u^1 .$$

We need to determine

$$E_s\ r_k = e^{-\kappa\cdot k}\cdot r_0 + \int_0^k e^{-\kappa\cdot(k-u)}\rho(u)du \quad Var\ r_k = \delta_1^2\cdot\int_0^k e^{-2\cdot\kappa\cdot u}du = \delta_1^2\cdot\frac{1 - e^{-2\cdot\kappa\cdot k}}{2\kappa} \text{ and}$$

$$Cov(r_k, R_k) = \frac{\delta_1^2}{\kappa}\cdot\int_0^k \left(1 - e^{-\kappa\cdot u}\right)\cdot e^{-\kappa\cdot u}du = \frac{\delta_1^2}{\kappa}\cdot\left(\frac{1}{2\kappa} - \frac{e^{-2\cdot\kappa\cdot k}\cdot(-1 + 2e^{\kappa\cdot k})}{2\kappa}\right),$$

where the index s denotes the condition $r_0 = s$.

Finally we denote $\Delta R_k = R_k - R_{k-1}$ and on the condition that $r_{k-1} = y$, we determine

$$E_y(R_k - R_{k-1}) = E\int_{k-1}^{k} r_u du = \frac{1}{\kappa}\cdot\left(1-e^{-\kappa}\right)\cdot y + \frac{1}{\kappa}\cdot\int_{k-1}^{k}\left(1-e^{-\kappa\cdot(k-z)}\right)\cdot\rho(z)dz,$$

$$Var_y(R_k - R_{k-1}) = Var\int_{k-1}^{k} r_u du = \frac{\delta_1^2}{\kappa^2}\cdot\int_{k-1}^{k}\left(1-e^{-\kappa\cdot(k-z)}\right)^2 dz = \frac{\delta_1^2}{\kappa^2}\cdot\left(-\frac{3-4e^{-\kappa}+e^{-2\kappa}-2\kappa}{2\kappa}\right),$$

$$Cov_y(r_k, R_k - R_{k-1}) = E\left(\delta_1\cdot\int_{k-1}^{k} e^{-\kappa\cdot(k-z)}dW_z^1\right)\cdot\left(\frac{\delta_1}{\kappa}\int_{k-1}^{k}(1-e^{-\kappa\cdot(k-z)})dW_z^1\right) =$$

$$= \frac{\delta_1^2}{\kappa}\cdot\left(\frac{1}{2\kappa} - \frac{e^{-2\cdot\kappa}\cdot(-1+2e^{\kappa})}{2\kappa}\right).$$

2.4 Actually achieved rate of return on investment

Denote J_k a random variable representing the actually achieved rate of return in the k^{th} policy year. We assume that

$$(5)\quad 1+J_k = e^{q\cdot\int_{k-1}^{k} r_u du + (1-q)\cdot\left[\nu + \delta_2\cdot\left(W_k^2 - W_{k-1}^2\right)\right]}.$$

It means that $1+J_k$ consists of the risk free yield, a risk premium ν (the value of risk) and a noise component, which can be modelled by increments of the second Wiener process $\{W_t^2, t\geq 0\}$, independent of $\{W_t^1, t\geq 0\}$. We can use this component for modelling an investment into non-risk free assets or imperfections of the market. We use a parameter q for modelling a mixed portfolio. We assume that $q\neq 0$.

When we need to consider an adjustment by a market value margin according to the fair value principle, we replace ν with $\nu' = \nu - \tau$. Thus

$$1+J_k' = e^{q\cdot\int_{k-1}^{k} r_u du + (1-q)\cdot\left[\nu' + \delta_2\cdot\left(W_k^2 - W_{k-1}^2\right)\right]}.$$

The distribution of $(1+J_k')$ is $\log N(q\cdot\mu_k + (1-q)\cdot\nu', q^2\cdot{\sigma_k}^2 + (1-q)^2\cdot\delta_2^2)$, where $\mu_k = E_y\Delta R_k$, $\sigma_k^2 = Var\ \Delta R_k$.

3 VALUE OF THE LIABILITY

We will apply the above-mentioned model to an endowment contract with profit sharing. Let K denote sum assured, ${}_kV_x$ reserve of premium for 1 unit at the end of the k^{th} policy year , n duration of insurance, i' technical interest rate, $s_{k,n}$ probability of surrender during the k^{th} policy year, ${}_k p_x = {}_{k-1}p_x \cdot (1 - q_{x+k-1} - s_{k-1,n})$ probability, that the policy is in force at the end of the k^{th} policy year.

3.1 Valuation of the liability using a diffusion process

For a bonus reserve at the end of the k^{th} policy year we assume the following formula
$Q_k = 0{,}8 \cdot (J'_k - i')_+ \cdot {}_{k-1}V_x + \left(1 + i' + 0{,}8 \cdot (J'_k - i')_+\right) \cdot Q_{k-1}$, for $k = 2,\dots,n$ and $Q_1 = 0$.
We assume that the technical interest on the bonus reserve Q_k is guaranteed as well as on the reserve of premium ${}_kV_x$.
For a valuation of the liability we have to consider a present value of expected future benefits. Therefore the expected present value of the liability can be expressed as

$$
\begin{aligned}
C = & -K \cdot {}_n p_x \cdot Ev^{(n)} \left(1 + Q_n\right) + \\
& -K \cdot \sum_{k=0}^{n-1} {}_k p_x \cdot s_{k+1,n} \cdot E\, v^{(k+1)} \cdot \left({}_{k+1}V_x^{storno} + 0{,}9 \cdot Q_k\right) - \\
& -K \cdot \sum_{k=0}^{n-1} {}_k p_x \cdot q_{x+k} \cdot Ev^{(k+1)} \cdot \left(1 + Q_k\right).
\end{aligned} \tag{6}
$$

It is clear that $Ev^{(k+1)} = e^{-E_s R_{k+1} + \frac{1}{2} VarR_{k+1}}$. To determine $Ev^{(k+1)} \cdot Q_k$ (which will be given later by the integral (11)) we will apply a recursive formula

$$
\begin{aligned}
E_s^z v^{(k)} Q_k = & \; 0{,}8 \cdot {}_{k-1}V_x \cdot E_s^z (J'_k - i')_+ \cdot (1 + I_k)^{-1} \cdot v^{(k-1)} + \\
& + (1 + i') \cdot E_s^z v^{(k)} \cdot Q_{k-1} + 0{,}8 \cdot E_s^z (J'_k - i')_+ \cdot (1 + I_k)^{-1} \cdot v^{(k-1)} \cdot Q_{k-1},
\end{aligned} \tag{7}
$$

where indices s and z denote conditions $r_0 = s, r_k = z$.
Utilizing the Markovian character of diffusion process $\{r_t\}$ and assuming $r_{k-1} = y$ we can write

$$(8)\quad E_s^z(J_k'-i')_+(1+I_k)^{-1}v^{(k-1)}=\int_{-\infty}^{\infty}E_y^z(J_k'-i')_+(1+I_k)^{-1}\cdot E_s^y v^{(k-1)}\cdot\frac{1}{\sqrt{2\pi\cdot Var\, r_{k-1}}}\cdot e^{-\frac{(y-E_s r_{k-1})^2}{2\cdot Var\, r_{k-1}}}dy,$$

$$(9)\quad \begin{aligned}&E_s^z(J_k'-i')_+(1+I_k)^{-1}v^{(k-1)}\cdot Q_{k-1}=\\&=\int_{-\infty}^{\infty}E_y^z(J_k'-i')_+(1+I_k)^{-1}\cdot E_s^y v^{(k-1)}\cdot Q_{k-1}\cdot\frac{1}{\sqrt{2\pi\cdot Var\, r_{k-1}}}\cdot e^{-\frac{(y-E_s r_{k-1})^2}{2\cdot Var\, r_{k-1}}}dy,\end{aligned}$$

$$(10)\qquad E_s^z v^{(k)}\cdot Q_{k-1}=\int_{-\infty}^{\infty}e^{-E_y^z\Delta R_k+\frac{1}{2}Var^z\Delta R_k}\cdot E_s^y v^{(k-1)}\cdot Q_{k-1}\cdot\frac{1}{\sqrt{2\pi\cdot Var\, r_{k-1}}}\cdot e^{-\frac{(y-E_s r_{k-1})^2}{2\cdot Var\, r_{k-1}}}dy.$$

While we assume $r_{k-1}=y$, the past and the future of the process are independent. Therefore we can integrate the product of mean value contingent by $r_{k-1}=y, r_k=z$ and mean value contingent by $r_{k-1}=y$, $r_0=s$.

For an evaluation of integrals (8), (9) we need to determine

$$\begin{aligned}&E_y^z(J_k'-i')_+\cdot(1+I_k)^{-1}=E_y^z(1+I_k)^{-1}\cdot\left((1+J_k')-(1+i')\right)_+=\\&=E\int_{q\cdot x+U>\ln(1+i')}e^{-x}\left(e^{q\cdot x+U}-(1+i')\right)\cdot\frac{1}{\sqrt{2\pi}\cdot\sigma_k}\cdot e^{-\frac{(x-\mu_k)^2}{2\cdot\sigma_k^2}}dx,\end{aligned}$$

where $U=(1-q)\cdot\left(v'+\delta_2\cdot\left(W_k^2-W_{k-1}^2\right)\right)$ has the distribution $N\left((1-q)\cdot v',(1-q)^2\cdot\delta_2^2\right)$. It is

$$\mu_k=E_y\Delta R_k+\frac{Cov(r_k,\Delta R_k)}{Var\, r_k}\cdot(z-E_y r_k)\quad\text{and}\quad\sigma_k^{\,2}=\left(1-\frac{Cov(r_k,\Delta R_k)^2}{Var\, r_k\cdot Var\,\Delta R_k}\right)\cdot Var\,\Delta R_k,$$

because the contingent distribution of R_k-R_{k-1} on conditions that $r_{k-1}=y, r_k=z$ is

$$N\left(E_y\Delta R_k+\frac{Cov_y(r_k,\Delta R_k)}{Var_y\, r_k}\cdot(z-E_y r_k),\left(1-\frac{Cov_y(r_k,\Delta R_k)^2}{Var_y\, r_k\cdot Var_y\,\Delta R_k}\right)\cdot Var_y\,\Delta R_k\right).$$

Let's assume that $U=u$, then

$$\begin{aligned}&E_y^z(J_k'-i')_+\cdot(1+I_k)^{-1}=\\&=e^{u+\frac{(q-1)\cdot\left(2\cdot\mu+\sigma^2\cdot(q-1)\right)}{2}}\cdot\Phi\left(-\frac{\frac{\ln(1+i')-u}{q}-\left(\mu_k+\sigma^2\cdot(1-q)\right)}{\sigma_k}\right)-\\&-(1+i')\cdot e^{-\mu_k+\frac{\sigma_k^2}{2}}\cdot\Phi\left(-\frac{\frac{\ln(1+i')-u}{q}-\left(\mu_k-\sigma_k^2\right)}{\sigma_k}\right)=g_k(u).\end{aligned}$$

Further we will integrate $\int_{-\infty}^{\infty} g_k(u) \cdot \frac{1}{\sqrt{2\pi} \cdot \delta_2} \cdot e^{\frac{-(u-v')^2}{2 \cdot \delta_2^2}} du$ using a numerical method which will be mentioned in the Paragraph 3.3 .

Similarly $E_s^y v^{(k-1)} = e^{-E_s^y R_{k-1} + \frac{1}{2} Var^y R_{k-1}}$,

where $E_s^y R_{k-1} = E_s R_{k-1} + \frac{Cov(r_{k-1}, R_{k-1})}{Var\ r_{k-1}} \cdot (y - E_s r_{k-1})$

and $Var^y R_{k-1} = \left(1 - \frac{Cov(r_{k-1}, R_{k-1})^2}{Var\ r_{k-1} \cdot Var\ R_{k-1}}\right) \cdot Var\ R_{k-1}$.

For an evaluation of integrals (9) and (10) we need the value of $E_s^y v^{(k-1)} \cdot Q_{k-1}$, which is given by a table of results from the previous step of the recursive formula (7).

Finally we have

$$(11) \qquad E_s v^{(k+1)} \cdot Q_k = \int_{-\infty}^{\infty} e^{-E_z \Delta R_k + \frac{1}{2} Var \Delta R_k} E_s^z v^{(k)} \cdot Q_k \cdot \frac{1}{\sqrt{2\pi \cdot Var\ r_k}} \cdot e^{-\frac{(z - E_s r_k)^2}{2 \cdot Var\ r_k}} dz \, .$$

The method described above is based on Markovian character of diffusion processes and therefore it can be used also with another diffusion interest rate model, for example Vašíček or Ho-Lee model.

Especially Vašíček model is suitable for easier and simpler computations, because its parameters are constant. Its mean reversion is typical for interest rates behaviour. It can be described by the equation

$$(12) \qquad d(r_t - \rho) = -\kappa (r_t - \rho) \cdot dt + \delta \cdot dW_t , \; t \in [0, T] .$$

The method doesn't use any approximations as well as it can be easily used for practical computation, for example using MS Excel. These can be consider as an important advantage of the method.

3.2 Market value margin

According to the principle stated in the Paragraph 1.2 we will adjust the variable J_k replacing v by $v' = v - \tau$ to get more conservative result.

The value of the parameter τ can be set from the equation $C = K \cdot \sum_{k=0}^{n-1} {}_k p_x \cdot E v^{(k)} \cdot \Pi_k$, where Π_k is a market premium at the primary insurance market and C is a function of the variable τ. In this case the parameter τ represents the market value of the risk, because policyholders with just this assessment of the risk will buy the contract at the primary insurance marker for the premium Π_k.

3.3 Numerical valuation

It is necessary to use a numerical method to calculate the integrals deducted in chapter 3.1. The method can be implemented in MS Excel, using e.g. Gauss's formula to evaluate the integrals.

Integrals (8), (9), (10) and (11) has the form $\int_{-\infty}^{\infty} g(y) \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy$. We will transform them to the form $\int_0^1 g\left(\Phi^{-1}(u) \cdot \sigma + \mu\right) du$ to solve the problem of integration interval $(-\infty, \infty)$.

3.4 Example

It is quite easy to construct a computing model in MS Excel, which is a significant advantage of the method described above. In the following table, we can find illustrative results for endowment contract with profit sharing with sum assured 100 000 and duration 10 years. Parameters of the Hull-White model in this example are $\kappa = 0{,}95$, $\delta_1 = 0{,}015$, $r_0 = 0{,}028$. Furthermore we assume $q = 80\%$, $\nu = \ln(1 + 0{,}08)$ a $\delta_2 = 0{,}12$.

k	Discounted death benefit	Discounted surrender benefit	Discounted maturity benefit	Total value of the liability C
1	126	632	0	758
2	119	1 073	0	1 192
3	121	1 364	0	1 485
4	120	1 536	0	1 656
5	119	1 616	0	1 735
6	120	1 627	0	1 747
7	124	1 585	0	1 709
8	126	1 503	0	1 629
9	127	1 392	0	1 519
10	135	1 259	35 758	37 151
Total	1 238	13 586	35 758	50 582

Table No.1: The modelled value of the liability.

4 LIABILITY ADEQUACY TESTING

The liability adequacy testing is an actual topic from the actuarial praxis. In the Czech Republic it is necessary to cope with a disharmony of Act No.363/1999 Coll.-Insurance Act which requires using the same statistical data and the same interest rate for calculation of premium and calculation of provision for life assurance (reserve of premium) and Act No.353/2001 Coll. which requires the fair value valuation of insurer's liability.

We can use a special case of the described model for liability adequacy testing, which will meet the criteria given by guidance No.3 of the Czech Society of Actuaries [6]. These criteria are mainly

- using a risk free yield curve for discounting cash flows,
- involving cash flows arising from embedded derivatives,
- using best estimates assumptions adjusted by provision for adverse deviation (PAD).

The recent discussion of IFRS working party of the Czech Society of Actuaries about the utilization of stochastic models resulted to opinion that risk free yield curve adjusted by PAD (0,25% is recommended) should be used for discounting . While embedded derivatives are assessed by a stochastic model, PAD should not be applied.

A risk free yield curve without any adjustment should be used for the allocation of profit sharing.

4.1 Special case of the model

Let assume that $q = 1$ in formula (5). Then $1 + J_k = e^{\int_{k-1}^{k} r_u du} = e^{(R_k - R_{k-1})} = 1 + I_k$.

Furthermore we can use already mentioned Vašíček model for the instantaneous risk free rate of interest r_t . By solving the equation (12) we get

$$r_k = \rho + (s - \rho) \cdot e^{-\kappa \cdot k} + \delta \cdot \int_0^k e^{-\kappa \cdot (k-u)} dW_u \text{ , where } r_0 = s \text{ .}$$

The variable R_k is given by the formula $R_k = \rho \cdot k + \frac{s - \rho}{\kappa}(1 - e^{-\kappa \cdot k}) + \delta \cdot \int_0^k \int_u^k e^{-\kappa \cdot (t-u)} dt \, dW_u$.

Necessary characteristics of r_k and R_k are given below, we assume the condition $r_0 = s$.

$$E_s r_k = \rho + (s-\rho)\cdot e^{-\kappa\cdot k}, Var\, r_k = \delta^2 \int_0^k e^{-2\cdot\kappa\cdot u} du = \delta^2 \frac{1-e^{-2\cdot\kappa\cdot k}}{2\cdot\kappa},$$

$$E_s R_k = \rho\cdot k + \frac{s-\rho}{\kappa}(1-e^{-\kappa\cdot k}),$$

$$Var\, R_k = \frac{\delta^2}{\kappa^2}\int_0^k (1-e^{-\kappa\cdot u})^2 du = \frac{\delta^2}{\kappa^2}\left(\frac{e^{-2\cdot\kappa\cdot k}\cdot(4e^{\kappa\cdot k} + 2e^{2\cdot\kappa\cdot k}\cdot\kappa\cdot k - 1) - 3}{2\cdot\kappa}\right) \text{ and}$$

$$Cov(r_k, R_k) = \frac{\delta^2}{\kappa}\int_0^k e^{-\kappa\cdot x}(1-e^{-\kappa\cdot u})du = \frac{\delta^2}{\kappa}\left(\frac{1-e^{-2\cdot\kappa\cdot k}\cdot(2e^{\kappa\cdot k}-1)}{2\cdot\kappa}\right).$$

The expected present value of the liability is

$$(13)\qquad \begin{aligned} C = & -K\cdot{}_n p_x \cdot Ev^{(n)}\left(1+Q_n\right) + \\ & -K\cdot\sum_{k=0}^{n-1} {}_k p_x \cdot E\, v^{(k+1)}\cdot\left(i' - I_{k+1}\right)_+ \cdot\left({}_k V_x + Q_k\right) - \\ & -K\cdot\sum_{k=0}^{n-1} {}_k p_x \cdot s_{k+1,n}\cdot E\, v^{(k+1)}\cdot\left({}_{k+1}V_x^{storno} + 0{,}9\cdot Q_k\right). \end{aligned}$$

For the allocation of profit sharing to the bonus reserve for $k = 2,...,n$ we have

$$Q_k = 0{,}8\cdot(I_k - i')_+ \cdot {}_{k-1}V_x + \left(1 + i' + 0{,}8\cdot(I_k - i')_+\right)\cdot Q_{k-1} \text{ and } Q_1 = 0.$$

Again we use the formula $Ev^{(k+1)} = e^{-E_s R_{k+1} + \frac{1}{2}VarR_{k+1}}$ and the recursive formula

$$(14)\qquad \begin{aligned} E_s^z v^{(k)} Q_k = & \; 0{,}8\cdot{}_{k-1}V_x \cdot E_s^z (I_k - i')_+ \cdot (1+I_k)^{-1}\cdot v^{(k-1)} + \\ & + (1+i')\cdot E_s^z v^{(k)}\cdot Q_{k-1} + 0{,}8\cdot E_s^z (I_k - i')_+ \cdot (1+I_k)^{-1}\cdot v^{(k-1)}\cdot Q_{k-1} \end{aligned}$$

for the determination of $Ev^{(k+1)}\cdot Q_k$.

We utilize the Markovian character, assuming $r_{k-1} = y$ we can write

$$(15)\qquad \begin{aligned} & E_s^z (I_k - i')_+ (1+I_k)^{-1} v^{(k-1)} = \\ & = \int_{-\infty}^{\infty} E_y^z (I_k - i')_+ (1+I_k)^{-1}\cdot E_s^y v^{(k-1)}\cdot \frac{1}{\sqrt{2\pi\cdot Var\, r_{k-1}}}\cdot e^{-\frac{(y-E_s r_{k-1})^2}{2\cdot Var r_{k-1}}} dy, \end{aligned}$$

$$(16)\qquad \begin{aligned} & E_s^z (I_k - i')_+ (1+I_k)^{-1} v^{(k-1)}\cdot Q_{k-1} = \\ & = \int_{-\infty}^{\infty} E_y^z (I_k - i')_+ (1+I_k)^{-1}\cdot E_s^y v^{(k-1)}\cdot Q_{k-1}\cdot \frac{1}{\sqrt{2\pi\cdot Var\, r_{k-1}}}\cdot e^{-\frac{(y-E_s r_{k-1})^2}{2\cdot Var r_{k-1}}} dy, \end{aligned}$$

$$(17)\qquad E_s^z v^{(k)}\cdot Q_{k-1} = \int_{-\infty}^{\infty} e^{-E_y^z R_1 + \frac{1}{2}Var^z R_1} E_s^y v^{(k-1)}\cdot Q_{k-1}\cdot \frac{1}{\sqrt{2\pi\cdot Var\, r_{k-1}}}\cdot e^{-\frac{(y-E_s r_{k-1})^2}{2\cdot Var r_{k-1}}} dy.$$

On the conditions that $r_{k-1}=y, r_k=z$ the distribution of R_k-R_{k-1} is

$$N\left(E_y R_1+\frac{Cov(r_1,R_1)}{Var\ r_1}\cdot(z-E_y r_1),\left(1-\frac{Cov(r_1,R_1)^2}{Var\ r_1\cdot Var\ R_1}\right)\cdot Var\ R_1\right).$$

Then $E_y^z\left((1+I_k)-(1+i')\right)_+\cdot(1+I_k)^{-1}=$

$$=E\int_{x>\ln(1+i')} e^{-x}\left(e^x-(1+i')\right)\cdot\frac{1}{\sqrt{2\pi}\cdot\sigma}\cdot e^{-\frac{(x-\mu)^2}{2\cdot\sigma^2}}dx=$$

$$=\Phi\left(-\frac{\ln(1+i')-\mu}{\sigma}\right)-(1+i')\cdot e^{-\mu+\frac{\sigma^2}{2}}\cdot\Phi\left(-\frac{\ln(1+i')-(\mu-\sigma^2)}{\sigma}\right),$$

where $\mu=E_y R_1+\frac{Cov(r_1,R_1)}{Var\ r_1}\cdot(z-E_y r_1)$ and $\sigma^2=\left(1-\frac{Cov(r_1,R_1)^2}{Var\ r_1\cdot Var\ R_1}\right)\cdot Var\ R_1$.

Similarly $E_s^y v^{(k-1)}=e^{-E_s^y R_{k-1}+\frac{1}{2}Var^y R_{k-1}}$, where $E_s^y R_{k-1}=E_s R_{k-1}+\frac{Cov(r_{k-1},R_{k-1})}{Var\ r_{k-1}}\cdot(y-E_s r_{k-1})$

and $Var^y R_{k-1}=\left(1-\frac{Cov(r_{k-1},R_{k-1})^2}{Var\ r_{k-1}\cdot Var\ R_{k-1}}\right)\cdot Var\ R_{k-1}$.

The value of $E_s^y v^{(k-1)}\cdot Q_{k-1}$ is the result of previous step of the recurrent formula (14).

Finally we determine $E_s v^{(k+1)}\cdot Q_k=\int_{-\infty}^{\infty} e^{-E_z R_1+\frac{1}{2}Var R_1} E_s^z v^{(k)}\cdot Q_k\cdot\frac{1}{\sqrt{2\pi\cdot Var\ r_k}}\cdot e^{-\frac{(z-E_s r_k)^2}{2\cdot Var\ r_k}}dz$

to calculate the formula (13).

4.2 Illustrative example

Let compare the value C calculated by the described stochastic model to result of a deterministic cash flow model. We again assume endowment contract with profit sharing, with sum assured 100 000 and duration 10 years. The technical interest rate is $i'=4\%$.

We use Vašíček model with parameters $\rho=0{,}06$, $r_0=0{,}024$, $\kappa=0{,}25$, $\delta=0{,}015$ in stochastic model.

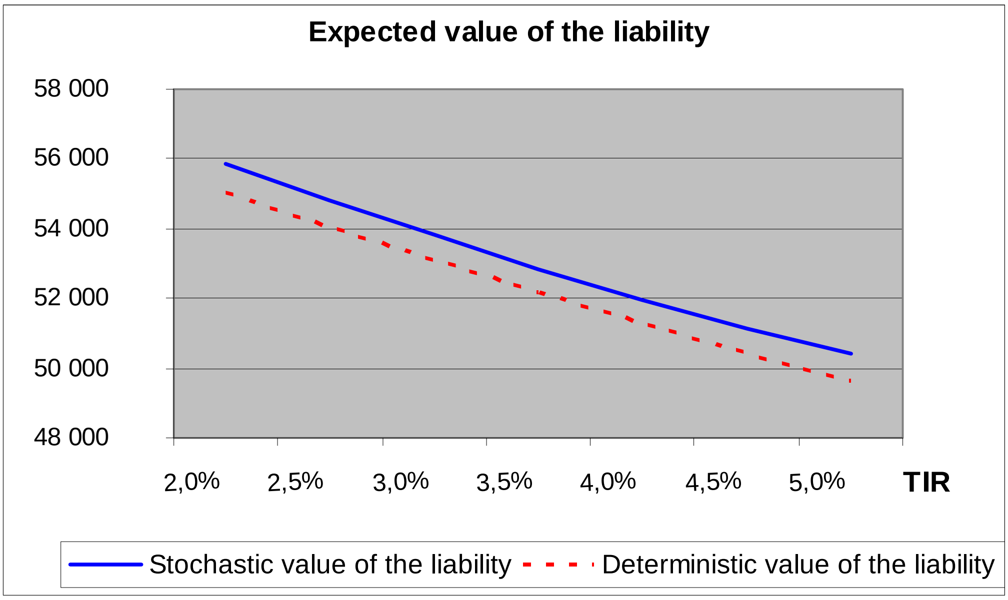
There is a deterministic risk free yield curve I_k^{Det} used in the deterministic model. For discounting there should be an adjustment (PAD) to risk free yield curve amounting to 0,25, as recommended in [6]. Therefore we use $I_k^{Det}-0{,}25\%$ for discounting. The decrement -0,25% applied to I_k^{Det} should be a simplified representation of the value of embedded derivatives which is already included in the stochastic model. Therefore we compare the value of the liability C calculated firstly using deterministic model without PAD, then including PAD and finally using the stochastic model.

Expected present value of liability C			
k	Deterministic model without PAD	Deterministic model including PAD	Stochastic model
1	772	774	772
2	1 214	1 220	1 215
3	1 507	1 518	1 510
4	1 674	1 690	1 683
5	1 748	1 769	1 763
6	1 755	1 780	1 777
7	1 713	1 742	1 742
8	1 631	1 663	1 666
9	1 520	1 553	1 561
10	36 689	37 576	38 259
Total	50 223	51 284	51 948

Table No.2: Comparison of stochastic and deterministic model.

We can see from the table above, that PAD applied to the deterministic calculation of C causes an increase of the liability amounted to 1 061. But the result is still less than the result of the stochastic calculation. This is especially caused by an undervaluation of the volatility risk.

Further we will show how the value of the liability C is affected by the technical interest rate i'. Other parameters are fixed.



Graph No.1: Dependence on a technical interest rate – comparison.

i'	Value of the liability C^{Stoch}	Value of the liability C^{Det}
2,0%	55 874	55 010
2,5%	54 811	54 053
3,0%	53 802	53 110
3,5%	52 841	52 180
4,0%	51 948	51 284
4,5%	51 151	50 434
5,0%	50 441	49 655

Table No. 3: Values used in graph No.1.

Graph No.1 shows, that the value of the liability C decreases with increasing value of i'. The reason is the decreasing value of the profit sharing credited to policyholders. The value obtained by the deterministic model including PAD is for the given values of i' still undervaluated compared to the result of the stochastic model..

5 CONCLUSION

The paper describes the method, how to use a diffusion model of an instantaneous rate of interest for discounting cash flows, using the Markovian character of diffusion processes. The described model is applied to valuation of the life insurance liability arising from an endowment contract with profit sharing. The method is universal and can be easily used for a practical computation even in MS Excel. Especially the application to adequacy liability testing can be useful in the actuarial practice.

The work is a part of the research project MSM0021620839 financed by MSMT.

6 BIBLIOGRAPHIES

[1] Ang A. and Sherris M., Interest rates risk management, North American Actuarial Journal, Vol. 1, Number 2, April 1997.

[2] Baxter M., Rennie A., Financial Calculus An Introduction to derivative pricing, Cambridge University Press, Cambridge 1996.

[3] Brigo, D.; Mercurio, F., Interest Rate Models Theory and Practice, Springer-Verlag, Berlin 2001.

[4] Mandl P., Pravděpodobnostní dynamické modely, Academia 1985.

[5] Fair Value of P&C Liabilities: Practical Implications: 3-5, The Casualty Actuarial Society, Arlington VA, 2004.

[6] Odborná směrnice č. 3, Test postačitelnosti technických rezerv životních pojištění, Česká společnost aktuárů, 2003, internetové stránky ČSpA, www.actuaria.cz.

[7] Report of solvency working party (Prepared for IAA Insurance Regulation Committee, February 2002).

[8] Zápis z 31. schůzky pracovní skupiny ČSpA pro IFRS, internetové stránky ČSpA, www.actuaria.cz.