- Title of the presentation: Models of the shot interest rate dynamics with θ - differentials.

- Surname- First name: Lappo Petr

- Organization: Belarusian State University

- Address: Faculty of Applied Mathematics & Computer Sciences, Belarusian State University, 4, Fr. Scorina av., Minsk-80, Belarus, 220080

- Telephone: + 375 17 209 51 29

- Fax: +375 17 2265548

- E-mail: LappoPM@bsu.by

- Abstract

Models of the shot interest rate dynamics with θ - differentials are considered in this paper. Stochastic differential equations with θ - differentials are in some sense generalization of the Ito equations, which are a special case when $\theta = 0$. The term structure equation is obtained and some concrete models are considered.

- Keywords

Shot interest rate dynamics model, θ-differential, stochastic differential equation, term structure equation, bond pricing.

INTRODUCTION

We consider a financial market in a probability space $(\Omega, F, P, \Im)$, where Ω is a set of elementary events, F is a σ-algebra of the events, P is a probability measure on F, and $\Im$ denotes the filtration. The main assets at this market are zero coupon bonds with different maturity dates T. A zero coupon bond with maturity date T, also called T-bond is a contract which guarantees the holder 1 unit of money to be paid on the date T. The price at time t of a bond with maturity date T is denoted by $p(t,T)$. We assume that

- there exists a market for T- bonds for every value of T,
- for every fixed T $\{p(t,T),\ 0 \le t \le T\}$ is a random process with the property $p(t,t) = 1$ for all t,
- for every fixed t the process $p(t,T)$ is continuously differentiable by the variable T.

The instantaneous forward rate at T, contracted at t is given by

$$f(t,T) = -\frac{\partial \ln p(t,T)}{\partial T}.$$

The instantaneous spot rate (shot interest rate) at the moment t is defined by the equality

$$r(t) = f(t,t).$$

We assume that there is a locally risk free asset V and its price process $V(t)$ satisfies the following equation

$$dV(t) = r(t)V(t)dt.$$

This equality holds with certainty. A natural interpretation of a riskless asset is that it corresponds to a bank with the short rate of interest r. This interest rate can be stochastic. In many models it is assumed that $r = r(t)$ is a continuous markovian process given apriori and it satisfies the Ito stochastic differential equation

$$dr(t) = \mu_r(t)dt + \sigma_r(t)dW(t), \qquad (1)$$

where $W(t)$ is a $\Im$ -adapted Wiener process and $\mu_r(t)$ and $\sigma_r(t)$ are enough smooth functions of the variables r and t. They are called the drift and the volatility of the instantaneous spot rate respectively. In the literature there is a large number of proposals how to specify the dynamics for r. We can write the equation (1) in the integral form

$$r(t) = r_0 + \int_0^t \mu_r(t)dt + (I)\int_0^t \sigma_r(t)dW(t), \qquad (2)$$

where the symbol (I) denotes the Ito integral. Using the Ito formula we can obtain that the price process satisfies the following stochastic differential equation (see, for example, Vasičec, (1977) or Bjork (1996))

$$dp = p\mu_P(t,T)dt - p\sigma_P(t,T)dW(t),$$

where

$$\mu_p(t,T) = \frac{1}{p(t,T,r)}\left[\frac{\partial}{\partial t} + \mu_r\frac{\partial}{\partial r} + \frac{1}{2}\sigma_r^2\frac{\partial^2}{\partial r^2}\right]p(t,T,r),$$

$$\sigma_P(t,T) = -\frac{1}{p(t,T,r)}\sigma_r\frac{\partial}{\partial r}p(t,T,r).$$

It is well known (Bjork (1996), Medvedev (2003)), that the bond price at the arbitrage free market satisfies the following partial differential equation

$$\frac{\partial p}{\partial t} + (\mu_r(t) + \lambda\sigma_r(t))\frac{\partial p}{\partial r} + \frac{1}{2}\sigma_r^2\frac{\partial^2 p}{\partial r^2} - rp = 0, \quad (3)$$

with the boundary condition

$$p(T,T,r) = 1\ . \quad (4)$$

In the equation (3)

$$\lambda = \lambda(t,r) = \frac{\mu_P(t,T) - r(t)}{\sigma_P(t,T)}$$

is the market price of risk. Usually the equation (3) is called the term structure equation. In order to be able to solve the term structure equation we must specify $\lambda, \mu,$ and σ exogenously. Below we present a list of the most popular models for r. If a parameter is time-dependent this is written out explicitly. Otherwise all parameters are constant.

1. Vasičec (Vasičec(1977))

$$dr(t) = (b - ar(t))dt + \sigma\, dW(t),$$

2. Cox-Ingersoll-Ross (CIR) (Cox, Ingersoll, Ross, 1985)

$$dr(t) = (b - ar(t))dt + \sigma\sqrt{r}(t)dW(t),$$

3. Black-Derman-Toy (Black, Derman, Toy, 1990)

$$dr(t) = a(t)r(t)dt + \sigma(t)r(t)dW(t),$$

4. Ho-Lee (Ho,Lee, 1986)

$$dr(t) = a(t)dt + \sigma\, dW(t),$$

5. Hull-White (extended Vasiček) (Hull, White, 1990)

$$dr(t) = (\Phi(t) - a(t)r(t))dt + \sigma(t)\,dW(t),$$

6. Hull-White (extended CIR)

$$dr(t) = (\Phi(t) - a(t)r(t))dt + \sigma(t)\sqrt{r(t)}\,dW(t),$$

7. Medvedev-Cox (M-C) (Medvedev,Cox,1996)

$$dr(t) = (ar(t) + \beta)dt + \sqrt{(\gamma r(t) + \delta}\,dW(t).$$

These models are discussed in Bjork (1996), and Medvedev (2003).

RESULTS

In this paper we assume that the dynamics of the shot rate is given by

$$r(t) = r_0 + \int_0^t \mu_r(s)ds + (\theta)\int_0^t \sigma_r(s)dW(s), \tag{5}$$

where the last term in (5) is θ-integral with $0 \le \theta \le 1$. The θ-integral is defined (see for details Pugachev, Sinitsyn (1998)) by

$$(\theta)\int_0^t q(s, r(s))dW(s) = \\ = l.i.m._{\substack{n\to\infty \\ \Delta\to 0}} \sum_{k=0}^{n-1} q(t_k, (1-\theta)r(t_k) + \theta\, r(t_{k+1}))\left[W(t_{k+1}) - W(t_k)\right], \tag{6}$$

where $0 = t_0 < t_1 < \ldots < t_n = t$, $\Delta = \max_{k=0,\ldots,n-1} (t_{k+1} - t_k)$ is the diameter of the subdivision, and the limit is considered in the square mean sense. The Ito integral is obtained from (6) if $\theta = 0$, the Stratanovich integral when $\theta = \frac{1}{2}$. The equation (5) could be written in the form of the stochastic θ-differential

$$d_\theta r = \mu_r(t)dt + \sigma_r(t)d_\theta W.$$

The relation between θ-integrals and Ito integrals is given by equality (Pugachev, Sinitsyn (1998))

$$(\theta)\int_{t_0}^t \psi(s, W(s))dW(s) = (I)\int_{t_0}^t \psi(s, W(s))dW(s) + \theta \cdot \int_{t_0}^t \frac{\partial\psi(s, W(s))}{\partial W} ds.$$

Considering the relation between θ -differentials and the Ito differentials and using ordinary technique (Bjork (1996)), we can found that the bond price on arbitrage free market satisfies the following partial differential equation

$$\frac{\partial p}{\partial t}+(\mu_r(t)+\lambda\sigma_r(t)+\theta\frac{\partial\sigma_r}{\partial r})\frac{\partial p}{\partial r}+\frac{1}{2}\sigma_r^2\frac{\partial^2 p}{\partial r^2}-rp=0\,, \tag{7}$$

with the boundary condition

$$p(T,T,r)=1\,,$$

and

$$\lambda(t,r)=\frac{\mu_P(t,T)-r(t)}{\sigma_P(t,T)}\,,$$

$$\mu_P(t,T)=\frac{1}{p(t,T,r)}\left[\frac{\partial}{\partial t}+\theta\left(\frac{\partial\sigma_r}{\partial r}\right)\frac{\partial}{\partial r}+\mu_r\frac{\partial}{\partial r}+\frac{1}{2}\sigma_r^2\frac{\partial^2}{\partial r^2}\right]p(t,T,r)\,,$$

$$\sigma_P(t,T)=-\frac{1}{p(t,T,r)}\sigma_r\frac{\partial}{\partial r}p(t,T,r)\,.$$

The equation (7) is the term structure equation when the shot rate dynamics follows the equation (5). In comparison with (3) we can see the additional term $\theta\frac{\partial\sigma_r}{\partial r}$ with $\frac{\partial p}{\partial r}$.

Let us consider some well known models of the shot rate dynamics where Ito differentials are substituted by θ -differentials. The Vasičec model with θ -differential does not give a new expression for the bond price because in this case

$$\frac{\partial\sigma_r}{\partial r}=0\,.$$

The Vasičec model and the CIR model with the Ito differentials are the special cases of the M-C model (Medvedev, 2003). We consider the M-C model with θ -differential.

$$d_\theta r(t)=(ar(t)+\beta)dt+\sqrt{(\gamma r(t)+\delta}\,d_\theta W(t)\,.$$

Using the relation between θ -differentials and Ito differentials we can write

$$dr(t)=(ar(t)+\beta+\frac{\theta\cdot\gamma}{2\sqrt{\gamma r(t)+\delta}})dt+\sqrt{(\gamma r(t)+\delta}\,dW(t)\,, \tag{8}$$

where the differential is the Ito differential. We will try to found the solution of the term structure equation in the form

$$P(t,T,r) = \exp\{A(t,T) - rB(t,T)\}. \tag{9}$$

In this situation the model is said to possess an affine term structure (Bjork ,1996). Applying the Ito formula to (8-9) we have the equation

$$\frac{\partial A(t,T)}{\partial t} - (1+\frac{\partial B(t,T)}{\partial t})r - (\alpha r + \beta + \frac{\theta\gamma}{2\sqrt{\gamma r+\delta}} + \lambda\sqrt{\gamma r+\delta})B(t,T) + \frac{1}{2}(\gamma r+\delta)B^2(t,T) = 0 \tag{10}$$

with the boundary conditions

$$A(T,T) = 0;\; B(T,T) = 0. \tag{11}$$

The equation (10) gives us the relations which must hold between $A, B, \alpha, \beta, \gamma, \delta$ and θ. To solve (10) in analytical form ($\theta \neq 0$) is a complex problem. We assume that $\lambda\sqrt{\gamma r+\delta} = \xi r + \eta$, where ξ and η are constants and consider an approximation of the solution. We make a Taylor expansion including first order terms

$$\frac{1}{\sqrt{\gamma r+\delta}} \approx \frac{1}{\sqrt{\delta}} - \frac{\gamma}{2\delta^{\frac{3}{2}}} r + \dots.$$

Then the approximation of the equation (10) is given by

$$\frac{\partial A(t,T)}{\partial t} - (1+\frac{\partial B(t,T)}{\partial t})r - (\alpha r + \beta + \xi r + \eta + \frac{\theta\gamma}{2\sqrt{\delta}} - \frac{\theta\gamma^2}{4\delta^{\frac{3}{2}}} r)B(t,T) + \frac{1}{2}(\gamma r+\delta)B^2(t,T) = 0$$

Here we use the same notations for the approximation and for the exact solution. Collecting coefficients in the r-term and in the other term we have

$$-1 - \frac{\partial B(t,T)}{\partial t} - (\alpha + \xi - \frac{\theta\gamma^2}{4\delta^{\frac{3}{2}}})B(t,T) + \frac{1}{2}\gamma B^2(t,T) = 0, \tag{12}$$

$$\frac{\partial A(t,T)}{\partial t} - (\beta + \eta + \frac{\theta\gamma}{2\sqrt{\delta}})B(t,T) + \frac{1}{2}\delta B^2(t,T) = 0 \tag{13}$$

The equation (12) is a Ricatti equation for the determination of $B(t,T)$ which does not involve $A(t,T)$. Having solved equation (12) we may then plug the solution $B(t,T)$ into equation (13) and simply integrate in order to obtain $A(t,T)$. The equation (12) can be solved, because it is a differential equation with separate variables. The solution is given by

$$B(t,T) = \frac{2(e^{\varepsilon(T-t)} - 1)}{2\varepsilon + \upsilon(e^{e(T-t)} - 1)}; \qquad \varepsilon = \sqrt{(\alpha - \frac{\theta\gamma^2}{4\delta^{3/2}} + \xi) + 2\gamma};$$

$$A(t,T) = -\frac{\delta}{\gamma}(T - t - B(t,T)) - \frac{\omega}{\gamma^2}\left[(\alpha - \frac{\theta\gamma^2}{4\delta^{3/2}} + \xi)(T-t) + \ln\left(\frac{\partial B(t,T)}{\partial t}\right)\right],$$

where we use the notations $\omega = (\alpha - \frac{\theta\gamma^2}{4\delta^{3/2}} + \xi)\delta - (\beta + \frac{\theta\gamma}{2\sqrt{\delta}} + \eta)\gamma$, $\upsilon = \varepsilon - (\alpha - \frac{\theta\gamma^2}{4\delta^{3/2}} + \xi)$. Of course, when $\theta = 0$, than we obtain the results from Medvedev (2003).

CONCLUSION

Thus we obtained the term structure equation when the shot rate dynamics is θ-differential. Also we have considered the M-C model with θ-differential and find some approximation of the bond price formula. It is expected that a presence of the additional parameter θ will allow to make one factor models more close to real yield data.

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