Accounting for Intervention Options: A Note on the Free Policy Reserve

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Abstract

The free policy reserve plays an important role in accounting for intervention options. We derive the dynamics of the statewise free policy reserve in the case of a constant interest rate. These dynamics have a simple interpretation. Hereby it is possible to write forward the free policy reserve and explain the changes in accounting and policy holder communications. We also introduce the notion of free policy surplus as the excess of assets over the free policy reserve. Finally, we indicate how the results generalize to dynamic interest rates.

Keywords: Free policy option, market valuation, differential equation, surplus, dynamic interest rates.

1 INTRODUCTION

The free policy option affects the concept of market valuation in life insurance. If the policy holder holds the option to stop premium payments against a decrease of future benefits, the insurance company should evaluate this option. The free policy option is somewhat similar in structure to the surrender option since both options are so-called intervention options. Upon intervention a valuation of future payments is performed and the result is used to resettle the future payments for the insurance contract. For the surrender option, the value is converted into an immediate surrender value pay-off. For the free policy option, the value is converted into a stream of future free policy benefits. Since the value used for resettlement is typically the so-called technical reserve, the intervention options are not free.

In Steffensen (2002), a general approach to intervention options in life insurance is presented. Steffensen (2002) suggests to apply optimal intervention theory which is essentially similar to optimal stopping theory. This theory appears in financial valuation in connection with e.g. American options and was introduced in life insurance by Grosen and Jørgensen (1997,2000) for valuation of the surrender option. In Steffensen (2002) the general setup is specialized in sections on the surrender and free policy options.

In a study of the free policy option, the free policy reserve appears as an important quantity. The free policy reserve at a time point is the market value of future payments given that the policy holder converts the policy into a free policy at that time point. The free policy reserve appears in the differential system which characterizes the market value of the contract including a free policy option, see Steffensen (2002). The free policy reserve is a lower bound for this market value, since the policy holder really can choose to convert immediately. Another lower bound is the market reserve for future payments disregarding the free policy option, since the policy holder really can choose never to convert.

In the Danish market value accounting standards, see Finanstilsynet (2003), the maximum of the two lower bounds mentioned in the previous paragraph is set aside as a market value. The market liability can be said to reflect a *now or never* conversion strategy of the policy holder since maximum is only taken over these two strategies. One can put up conditions on which the now or never value actually equals the market value, see Steffensen (2002) and Nielsen (2002). Thus, the free policy reserve is crucial in general, and in Danish market value accounting standards in particular.

The technical reserve is characterized by the celebrated Thiele differential equation. If the so-called valuation basis for market valuation is deterministic, the market reserve has a characterization which is similar in structure. Due to the piecewise constant stochastic processes underlying these quantities the differential equation also works as piecewise forward equations describing the dynamics of these reserves. However, the differential equation and the dynamics of the free policy reserve cannot be directly derived from a classical Thiele differential equation. It is the main object of this short note to derive the dynamics of the free policy option. This not a difficult mathematical task but just as important is the interpretation of the elements of these dynamics. We endow the reader with intuition about the terms.

An important notion in modern life insurance mathematics is the technical surplus. The excess of some notion of assets over the technical reserve and its dynamics are studied by Ramlau-Hansen (1988), Norberg (1999), and Steffensen (2004). All three authors relate such a technical surplus to the classical notion of surplus which was based on excess of technical reserve over market reserve. The systematic surplus contributions are the same for both notions of surplus. The future surplus contributions quantify the ability to pay out dividends to policy holders, possibly in terms of bonus payments. Studying the free policy reserve, it may be relevant to introduce the notion of free policy surplus as excess of assets over the free policy reserve. This will lead to free policy surplus contributions quantifying the ability, stemming from the reserves and not from future premiums, to pay out dividends to policy holders. In the Danish market value accounting standards, these quantities appear in the calculation of elements of the so-called individual bonus potential.

The starting point for interpreting the elements of the dynamics of the free policy reserve is market valuation under constant interest rates. However, a natural next step is to generalize to dynamic interest rates. The effect of dynamic interest rates on the free policy reserve is similar to the effect on market reserves. Since the main object of our study is the dynamics of the free policy reserve compared to the dynamics of the technical and market reserves, we shall just indicate these effects. Classical interest rate theory applies.

In addition to the literature already mentioned in this introduction, we shall refer to Linnemann (2003,2004) who also addresses the problem of valuation under a free policy option. Linnemann (2003,2004) works with a free policy reserve, there called a paid-up policy value, similar to ours. Since Linnemann works in a survival model and we here shall work with a general finite state Markov chain as a model for the policy, our results generalize certain results by Linnemann (2003,2004).

The outline of the note is as follows. In Section 2, we derive and interpret the dynamics of the free policy reserve. Furthermore we relate to these dynamics - and correct - a result in Steffensen (2002). In Section 3, we derive the contributions to the free policy surplus and compare with the contributions to the technical surplus. In Section 4, we indicate how the results in Section 2 generalize to dynamic interest rates.

2 THE FREE POLICY DIFFERENTIAL EQUATION

We consider an insurance policy issued at time 0 and terminating at a fixed finite time n. There is a finite set of states of the policy, $\mathcal{J} = \{0, \ldots, J\}$. Let $Z(t)$ denote the state of the policy at time $t \in [0, n]$, and define the associated indicator processes $(I^j)_{j \in \mathcal{J}}$ and counting processes $\left(N^{jk}\right)_{j,k \in \mathcal{J}}$, respectively, by $I^j(t) = 1[Z(t) = j]$ and $N^{jk}(t) = \#\{s \,|s \in (0, t], Z(s-) \neq j, Z(s) = k\}$.

Let $B(t)$ denote the total amount of contractual benefits less premiums payable during the time interval $[0, t]$. We assume that it develops in accordance with the dynamics

$$dB(t) = \sum_j I^j(t)\, dB^j(t) + \sum_{k\neq j} b^{jk}(t)\, dN^{jk}(t),$$

where B^j is a deterministic and sufficiently regular function specifying payments due during sojourns in state j, and b^{jk} is a deterministic and sufficiently regular function specifying payments due upon transition from state j to state k. We assume that each B^j decomposes into an absolutely continuous part and a discrete part, i.e.

$$dB^{Z(t)}(t) = b^{Z(t)}(t)\, dt + \Delta B^{Z(t)}(t),$$

where $\Delta B^j(t) = B^j(t) - B^j(t-)$, when different from 0, is a jump representing a lump sum payable at time t if the policy is then in state j. The set of time points with jumps in $(B^j)_{j\in\mathcal{J}}$ is $\mathcal{D} = \{t_0, t_1, \ldots t_q\}$ where $t_0 = 0$ and $t_q = n$. Let $B^+(t)$ denote the total amount of benefits payable during the time interval $[0,t]$, i.e. $dB^+(t) = \max(dB(t), 0)$, and introduce the corresponding notation for the element of this process, i.e. $dB^{+j}(t) = \max(dB^j(t), 0)$, etc.

We assume that the insurance company works with a market valuation basis (r, μ). The interest rate r reflects the assumption on a constant market interest rate. The intensities $\mu = \left(\mu^{jk}\right)_{j,k\in\mathcal{J}}$ reflects the assumption that the process Z under market valuation is assumed to be a time-continuous Markov process on the state space $\mathcal{J}$ such that N^{jk} admits the stochastic intensity process $\left\{I^j(t-)\,\mu^{jk}(t)\right\}_{t\in[0,n]}$, i.e. such that a martingale is constituted by

$$M^{jk}(t) = N^{jk}(t) - \int_0^t I^j(s)\,\mu^{jk}(s)\, ds.$$

If the market is neutral with respect to risk in Z then the intensities for market valuation μ coincide with the objective intensities describing the objective probability distribution of Z. For technical valuations the insurance company is assumed to work with a technical valuation basis (r^*, μ^*).

We can now put up the *statewise market reserve* defined by

$$V^j(t) = E\left[\int_t^n e^{-r(s-t)} dB(s) \,\middle|\, Z(t) = j\right]. \tag{1}$$

It is well-known that the dynamics of the statewise market reserve is given by

$$\begin{aligned} dV^j(t) &= rV^j(t)\, dt - dB^j(t) - \sum_{k\neq j} \mu^{jk}(t)\, R^{jk}(t)\, dt, \qquad (2)\\ V^j(n) &= 0, \end{aligned}$$

where R^{jk} is the risk sum process defined by

$$R^{jk}(t) = b^{jk}(t) + V^k(t) - V^j(t). \tag{3}$$

Based on (1) and (2) we are now ready to introduce three additional notions of reserves. The *statewise market benefit reserve* denoted by V^{+j} is obtained by replacing B by B^+ in (1). The dynamics of the statewise benefit reserve is now obtained by replacing $\left(V^{j}, R^{jk}, B^{j}, b^{jk}\right)$ by $\left(V^{+j}, R^{+jk}, B^{+j}, b^{+jk}\right)$ in (2) and (3). The *statewise technical reserve* denoted by V^{*j} is obtained by replacing (r, E) by (r^{*}, E^{*}) in (1), where E^{*} denotes that expectation is taken with respect to the distribution assumptions in the technical basis. The dynamics of the statewise technical reserve is obtained by replacing $\left(V^{j}, R^{jk}, r, \mu^{jk}\right)$ by $\left(V^{*j}, R^{*jk}, r^{*}, \mu^{*jk}\right)$ in (2) and (3). The *statewise technical benefit reserve* denoted by V^{*+j} is obtained by replacing (B, r, E) by (B^{+}, r^{*}, E^{*}) in (1). The dynamics of the statewise technical benefit reserve is obtained by replacing $\left(V^{j}, R^{jk}, B^{j}, b^{jk}, r, \mu^{jk}\right)$ by $\left(V^{+*j}, R^{+*jk}, B^{+j}, b^{+jk}, r^{*}, \mu^{*jk}\right)$ in (2) and (3).

We assume that the policy holder can convert the policy into a free policy, i.e. a policy which continues without premiums but with decreased benefits. The decrease of benefits is proportional in the sense that there exists a free policy factor f, such that the free policy benefits are given by the stream of benefits $f^{j}(t)\, dB^{+}(s)$, $t < s \leq n$, given that the policy is in state j upon conversion at time t. The free policy factor is determined such that the technical reserve before and after conversion are equal, i.e.

$$\begin{aligned} V^{*j}(t) &= E^{*}\left[\int_{t}^{n} e^{-r^{*}(s-t)} f^{Z(t)}(t)\, dB^{+}(s) \middle| Z(t) = j\right] \Leftrightarrow \\ f^{j}(t) &= \frac{V^{*j}(t)}{V^{+*j}(t)}. \end{aligned} \tag{4}$$

We can now introduce the object of our study, the *statewise free policy reserve* given by

$$\begin{aligned} V^{fj}(t) &= E\left[\int_{t}^{n} e^{-r(s-t)} f^{Z(t)}(t)\, dB^{+}(s) \middle| Z(t) = j\right] \\ &= f^{j}(t)\, V^{+j}(t) \\ &= \frac{V^{*j}(t)}{V^{+*j}(t)} V^{+j}(t). \end{aligned}$$

For notational convenience we choose not to show the dependence of f and V^{f} on the technical basis in the notation. The statewise free policy reserve is a market reserve for the free policy payments. This differs from the statewise market reserve V^{j} to the extent that the market basis (r, μ) differs from the technical basis (r^{*}, μ^{*}). Only if the two bases equate we have that $V^{fj}(t) = f^{j}(t)\, V^{+j}(t) = \left(V^{j}(t) / V^{+j}(t)\right) V^{+j}(t) = V^{j}(t)$. We are interested in deriving the dynamics of the statewise free policy reserve.

Firstly, we calculate, based on the product rule $dV^{*j}(t) = f^{j}(t-)\, dV^{+*j}(t) + V^{+*j}(t)\, df^{j}(t)$, a term $dA^{j}(t)$ which we will speak of as a correction term,

$$\begin{aligned} dA^{j}(t) &= V^{+*j}(t)\, df^{j}(t) \\ &= dV^{*j}(t) - f^{j}(t-)\, dV^{+*j}(t) \\ &= -dB^{j}(t) - \textstyle\sum_{k \neq j} \mu^{*jk}(t)\, R^{*jk} dt \\ &\quad + f^{j}(t-)\left(dB^{+j}(t) + \textstyle\sum_{k \neq j} \mu^{*jk}(t)\, R^{+*jk}(t)\, dt\right). \end{aligned}$$

The part of the correction term $f^{j}(t-)\left(dB^{+j}(t) + \sum_{k \neq j} \mu^{*jk}(t)\, R^{+*jk}(t)\, dt\right)$ equals minus the cash flow on the technical reserve over $(t, t + dt]$ had the policy been converted to free policy at time t. The part of the correction term $-dB^j(t) - \sum_{k\neq j}\mu^{*jk}(t)\, R^{*jk} dt$ equals the cash flow on the technical reserve over $(t, t+dt]$ given that the policy is not converted to free policy at time t. Thus, the correction term can be interpreted as the adjustment to the technical reserve, following from *not converting to free policy after all.*

Using the product rule $dV^{fj}(t) = f^j(t-)\, dV^{+j}(t) + V^{+j}(t)\, df^j(t)$, we can now derive the dynamics of the free policy reserve which is the main result of this note from a mathematical point of view,

$$\begin{aligned} dV^{fj}(t) &= rV^{fj}(t)\, dt - f^j(t-)\left(dB^{+j}(t) + \sum\nolimits_{k\neq j}\mu^{jk}(t)\, R^{+jk}(t)\, dt\right) \\ &\quad + \frac{V^{+j}(t)}{V^{+*j}(t)} dA^j(t). \end{aligned} \tag{5}$$

These dynamics have an obvious interpretation. The free policy reserve changes according to the market basis and according to the free policy benefits as if the policy actually had been converted at time t. This gives rise to the first part of the statewise free policy reserve increment $rV^{fj}(t)\, dt - f^j(t-)\, (dB^{+j}(t) + \sum_{k\neq j}\mu^{jk}(t)\, R^{+jk}(t)\, dt)$. However, the policy was not converted for which reason the correction term comes into play. The correction term $dA^j(t)$ was interpreted as the adjustment to the technical reserve, following from not converting. It should be considered as a single premium over $(t, t+dt]$. This single premium trades into the proportion $dA^j(t)/V^{+*j}(t)$ of future benefits and the market value of this stream of benefits equals $V^{+j}(t)\, dA^j(t)/V^{+*j}(t)$. Thus, the second part of the statewise free policy reserve increment is the market value of the correction terms $dA^j(t)$.

Example 1 *Consider a survival model with an endowment insurance with sum insured equal to* 1 *and premium rate* π. *We condition all quantities on the insured being alive. Then the four elementary notions of reserves and the free policy factor become*

$$\begin{aligned} V(t) &= \int_t^n e^{-\int_t^s r+\mu}(\mu(s) - \pi)\, ds + e^{-\int_t^n r+\mu}, \\ V^{+}(t) &= \int_t^n e^{-\int_t^s r+\mu}\mu(s)\, ds + e^{-\int_t^n r+\mu}, \\ V^{*}(t) &= \int_t^n e^{-\int_t^s r^*+\mu^*}(\mu^*(s) - \pi)\, ds + e^{-\int_t^n r^*+\mu^*}, \\ V^{+*}(t) &= \int_t^n e^{-\int_t^s r^*+\mu^*}\mu^*(s)\, ds + e^{-\int_t^n r^*+\mu^*}, \\ f(t) &= \frac{V^*(t)}{V^{+*}(t)}. \end{aligned}$$

The correction term $dA(t)$ *becomes*

$$\begin{aligned} dA(t) &= \pi - \mu^*(t)\,(1 - V^*(t)) + f^*(t)\,\mu^*(t)\left(1 - V^{+*}(t)\right) \\ &= \pi - \mu^*(t)\,(1 - f^*(t)). \end{aligned}$$

Finally we can write down the dynamics of the free policy reserve,

$$\frac{d}{dt}V^{f}(t) = (r+\mu(t))\,V^{f}(t)\,dt - f(t)\,\mu(t) + \frac{V^{+}(t)}{V^{+*}(t)}\left(\pi - \mu^{*}(t)\,(1-f(t))\right).$$

Since the interpretation of these dynamics plays an important role in this note we choose to repeat it here for the example. The first part of $dV^{f}(t)$ is a forward writing corresponding to (2) as if the contract was converted into a free policy at time t. In the last part the technical reserve is adjusted according to the premium paid after all. The natural premium for the free policy benefit, $\mu^{*}(t)\,f(t)$, is reinjected together with the premium and the natural premium for the full benefit, $\mu^{*}(t)$, is withdrawn instead. Multiplication by the factor $V^{+}(t)/V^{+*}(t)$ gives the market value of the adjustment to the technical reserve.

We end this section by comparing our derived dynamics with the results in Steffensen (2002). In there, a more general free policy reserve was introduced. Let the free policy reserve $V^{fij}(t,u)$ be the market value of future free policy benefits given that the policy is in state j and time t and given that the policy was in state i when it was converted into free policy at time $t-u$. Thus u is the duration since conversion. It is easily seen that this free policy reserve can be written, in terms of the notation introduced above, as

$$V^{fij}(t,u) = f^{i}(t-u)\,V^{+j}(t). \tag{6}$$

In Steffensen (2002, Equation (19)) a partial differential system for $V^{fij}(t,u)$ is given. Adding an updating relation holding for $t\in\mathcal{D}$ (Steffensen (2002) works in the special case $\mathcal{D}=\{n\}$), this differential system reads with our notation,

$$\begin{aligned}
\frac{\partial}{\partial t}V^{*fij}(t,u) &= rV^{fij}(t,u) - f^{i}(t-u)\,b^{+j}(t) \qquad (7)\\
&\quad -\textstyle\sum_{k\neq j}\mu^{jk}(t)\,R^{fijk}(t,u) - \frac{\partial}{\partial u}V^{fij}(t,u),\;\; t\notin\mathcal{D},\\
0 &= f^{i}(t-u)\,\Delta B^{+j}(t) + V^{fij}(t,u) - V^{fij}(t-,u),\; t\in\mathcal{D},\\
V^{ij}(n,u) &= 0,
\end{aligned}$$

where

$$R^{fijk}(t,u) = f^{i}(t-u)\,b^{+jk}(t) + V^{fik}(t,u) - V^{fij}(t,u).$$

Here we can calculate, by (4) and (6) and differentiation of a fraction,

$$\frac{\partial}{\partial u}V^{fij}(t,u) = -\frac{V^{+j}(t)}{V^{+*i}(t-u)}\left.\frac{d}{dt}A^{i}(t)\right|_{t-u}.$$

Then the differential system (7) on $(u,i)=(0,j)$ specializes into the ordinary differential system

$$\frac{d}{dt}V^{fj}(t) = rV^{fj}(t) - f^{j}(t)\,b^{+j}(t) \tag{8}$$

$$
\begin{aligned}
& -\sum_{k\neq j}\mu^{jk}(t)\,R^{fjk}(t)-\frac{V^{+j}(t)}{V^{+*j}(t)}\frac{d}{dt}A^{j}(t)\,,\quad t\notin\mathcal{D}\,,\\
0 \;=\;& f^{j}(t-)\left(\Delta B^{+j}(t)+V^{+j}(t)\right)-V^{fj}(t-)\,,\ t\in\mathcal{D},
\end{aligned}
$$

where

$$
R^{fjk}(t)=f^{j}(t)\left(b^{+jk}(t)+V^{+k}(t)\right)-V^{fj}(t)\,.
$$

But (8) characterizes exactly the same function as (5) and we have specialized (7) into (5). For $t\notin\mathcal{D}$ this is easily seen to hold while the conclusion for $t\in\mathcal{D}$ follows after some simple rearrangements.

3 THE FREE POLICY SURPLUS

The notions of surplus studied by Ramlau-Hansen (1988), Norberg (1999), and Steffensen (2004) share the basic idea of subtracting the technical surplus from the cash value of past payments including capital gains. Assuming that the payments are invested in a portfolio which earns return rate r on investments, we define the surplus process X^{*} by

$$
X^{*}(t)=-\int_{0-}^{t}e^{\int_{s}^{t}r}dB(s)-\sum_{j}I^{j}(t)\,V^{*j}(t)\,.
$$

Repeating the derivation of these surplus dynamics from Norberg (1999), we obtain

$$
\begin{aligned}
dX^{*}(t) &= -dB(t)+r\left(X^{*}(t)+\sum_{j}I^{j}(t)\,V^{*j}(t)\right)dt\\
&\quad -\sum_{j}I^{j}(t)\,dV^{*j}(t)-\sum_{k\neq j}\left(V^{*k}(t)-V^{*j}(t)\right)dN^{jk}(t)\\
&= rX^{*}(t)\,dt+\sum_{j}I^{j}(t)\,dC^{*j}(t)-\sum_{k\neq j}R^{*jk}(t)\,dM^{jk}(t)\,,
\end{aligned}
$$

where C^{*j} is the process of statewise accumulated surplus contributions given by

$$
dC^{*j}(t)=(r-r^{*})\,V^{*j}(t)\,dt+\sum_{k\neq j}\left(\mu^{*jk}(t)-\mu^{jk}(t)\right)R^{*jk}(t)\,dt.
$$

These surplus contributions reflect that past and future payments are evaluated differently. Past payments are evaluated on the basis of realized returns and expected transitions through (r,μ). Future payments are evaluated on the basis of technical returns and technically expected transitions (r^{*},μ^{*}). The martingale increments $R^{*jk}(t)\,dM^{jk}(t)$ reflect that realized transitions do not coincide with expected transitions.

Following the ideas of the notion of technical surplus, we now introduce a corresponding notion of free policy surplus. The free policy surplus is defined as the technical surplus, except for that the future payments are measured by the free policy reserve, i.e.

$$
X^{f}(t)=-\int_{0-}^{t}e^{\int_{s}^{t}r}dB(s)-\sum_{j}I^{j}(t)\,V^{fj}(t)\,.
$$

One should expect the structure of contributions to the free policy surplus to become essentially different from the structure of contributions to the technical surplus. The technical surplus contributions stem from two different valuation bases for the same payment process measured in the past and the future, respectively. The free policy surplus contributions stem from valuation of two different payment processes in the past and the future, respectively, but under the same basis. In the past we measure realized payments given no conversion into free policy and in the future we measure payments based on immediate conversion into free policy. Similar calculations as the ones leading to the dynamics of the technical surplus now gives the dynamics of the free policy surplus,

$$\begin{aligned} dX^{f}(t) &= -dB(t) + r(t)\left(X^{f}(t) + \sum\nolimits_{j} I^{j}(t) V^{fj}(t)\right) dt \\ &\quad - \sum\nolimits_{j} I^{j}(t)\, dV^{fj}(t) - \sum\nolimits_{k\neq j}\left(V^{fk}(t) - V^{fj}(t)\right) dN^{jk}(t) \\ &= r(t) X^{f}(t)\, dt + \sum\nolimits_{j} I^{j}(t)\, dC^{fj}(t) - R^{fjk}(t)\, dM^{jk}(t), \end{aligned}$$

where the processes of contributions and the risk sums are given by

$$\begin{aligned} dC^{fj}(t) &= f^{j}(t-)\left(dB^{+j}(t) + \sum\nolimits_{k\neq j}\mu^{jk}(t) R^{+jk} dt\right) - \frac{V^{+j}(t)}{V^{+*j}(t)} dA^{j}(t) \\ &\quad - dB^{j}(t) - \sum\nolimits_{k\neq j}\mu^{jk}(t) R^{fjk}(t)\, dt, \\ R^{fjk}(t) &= b^{jk}(t) + V^{fk}(t) - V^{fj}(t). \end{aligned} \tag{9}$$

These derivations are similar to the derivation leading to Steffensen (2002, Equation (20)). Steffensen (2002) works with a particular process called Y_u^t defined such that Y_t^0 equals the discounted surplus defined here, i.e. $Y_t^0 = \exp(-\int_0^t r) X^{*f}(t)$. The calculations above, however, disclose an error in Steffensen (2002). In Steffensen (2002, p 83_1), $f^{*j}(t) V^{+k}(t)$ and $f^{*k}(t) V^{+k}(t)$ are wrongly taken to equate such that the free policy surplus contributions specified in Steffensen (2002, p 84^6) differ from (9) by the term $\sum_{k\neq j}\mu^{jk}(t) V^{+k}(t)\left(f^{*k}(t) - f^{*j}(t)\right)$. This error does not violate the conclusions in Steffensen (2002, Section 6.1) which are worked out in a survival model. The conclusions in Steffensen (2002, Section 6.1) correspond to our findings in our continuation of Example 1 below.

A study of the contributions to the free policy surplus helps to find possibly explicit optimal free policy intervention strategies. If the surplus contributions are positive in all states and at all times, then it is optimal to convert into free policy immediately, and therefore the market value equals the free policy reserve. If the surplus contributions are negative in all states and at all times, then it is optimal never to convert and therefore the market value equals the market reserve. These conclusions follow from Steffensen (2002). In between these two extreme cases there are a continuum of situations. Some of them lead to explicit market values which are larger than the larger of the free policy reserve and the market reserve. Such explicit results are obtained by Nielsen (2002). It should be mentioned that these considerations are based on the payment process B exclusively. If there are additional options, like e.g. bonus options, the optimal strategies may change. However, the considerations still serve as natural fixed points in the general decomposition of liabilities.

Example 1 continued *For the insurance contract described in Example 1, we can derive assumptions which ensure that contributions to the free policy surplus are positive. After several rearrangements one reaches at*

$$\begin{aligned} dC^f(t) &= \frac{1}{V^{+*}(t)}\Big(V^{+*}(t) - V^*(t)\Big)\left(\mu^*(t) - \mu(t)\right) \\ &+ \frac{1}{V^{+*}(t)}\left(1 - \frac{V^+(t)}{V^{+*}(t)}\right)\Big(\pi V^{+*}(t) - \mu^*(t)\Big(V^{+*}(t) - V^*(t)\Big)\Big). \end{aligned}$$

If $\mu^*(t) \geq \mu(t)$ *and* $r^* \geq r$ *one easily realizes that* $1 - \frac{V^+(t)}{V^{+*}(t)} \geq 0$. *Since* $V^{+*}(t) - V^*(t) \geq 0$ *we can conclude that* $dC^f(t) \geq 0$ *if* $\pi V^{+*}(t) - \mu^*(t)\left(V^{+*}(t) - V^*(t)\right) \geq 0$. *But this is seen to hold if* μ^* *is increasing since*

$$\begin{aligned} &\pi V^{+*}(t) - \mu^*(t)\Big(V^{+*}(t) - V^*(t)\Big) \\ = &\ \pi \int_t^n e^{-\int_t^s r^* + \mu^*}\left(\mu^*(s) - \mu^*(t)\right) ds + \pi e^{-\int_t^n r^* + \mu^*}. \end{aligned}$$

In Example 1 the derivation of assumptions ensuring that the free policy surplus contributions are positive requires some calculations and rearrangements even though this is a very simply insurance contract. We invite the reader to search for the general assumptions on the relation between bases and on the payment process which ensure positive contributions in general.

4 DYNAMIC INTEREST RATES

In this section we indicate how the dynamics of V^{*j} and V^{*fj} can be generalized to dynamic market interest rates. Concerning V^{*fj} the only effect comes from the dynamics of $V^{+j}(t)$ since the free policy factor is unaffected. Noting that the dynamics of $V^{+j}(t)$ is similar in structure to the dynamics of $V^j(t)$, we therefore focus on generalizing the dynamics of V^j to dynamic interest rates. We leave it to the reader to derive similar results for V^{fj}.

The first approach to valuation under dynamic interest rates is given by a flat term structure, i.e. at time t the interest rate equals $r(t)$ over $(t, n]$. Although such a model of dynamic interest rates has problematic theoretical consequences, this is still, for many practical purposes, a simple way to start. Thus, we define the statewise market reserve as

$$V^j(t) = E\left[\int_t^n e^{-r(t)(s-t)} dB(s) \middle| Z(t) = j\right]. \tag{10}$$

Calculations similar to those leading to (2) result in the following reserve dynamics,

$$dV^j(t) = V^j(t)\, dG^j(t) - dB^j(t) - \textstyle\sum_{k \neq j} \mu^{jk}(t)\, R^{jk}(t)\, dt,$$

where the capital gain process $G^j(t)$ is given by

$$dG^j(t) = r(t)\, dt - D^j(t)\, dr(t).$$

Here, the statewise duration $D^j(t)$ is given by

$$D^j(t) = \frac{V^{jD}(t)}{V^j(t)} = -\frac{1}{V^j(t)} \frac{\partial}{\partial r(t)} V^j(t),$$

where $V^{jD}(t)$ is obtained by replacing $dB(s)$ in (10) by $(s-t)\,dB(s)$.

Parallel shifts in a flat term structure lead to arbitrage possibilities and therefore the formulas above are inconsistent with an assumption of no arbitrage, see e.g. Munk (2004). We therefore quickly wash our hands with soap and go on to implementation of a factor diffusion model. Introduce a factor Y following the dynamics

$$\begin{aligned} dY(t) &= \alpha(Y(t))\,dt + \beta(Y(t))\,dW(t), \\ Y(0) &= y_0, \end{aligned}$$

where W is a standard Brownian motion and where α and β are deterministic and sufficiently regular functions. We assume that the risk free interest rate at time t can be written as $r(Y(t))$. From market prices in the bond market one should now determine a market price process of risk $\eta(Y(t))$, assuming that this is only dependent on $Y(t)$. The dynamics of Y can now be written

$$dY(t) = (\alpha(Y(t)) - \beta(Y(t))\,\eta(Y(t)))\,dt + \beta(Y(t))\,dW^Q(t),$$

where W^Q is a standard Brownian motion under the valuation measure Q. Since this is not at all meant as an exposition on applications of interest rate models to life insurance, we shall not go into details here with how η, or equivalently Q, is determined. The statewise market reserve is then defined by

$$V^j(t,y) = E^Q\left[\int_t^n e^{-\int_t^s r(Y(\tau))d\tau} dB(s) \,\middle|\, Z(t) = j, Y(t) = y\right].$$

Here, with a slight abuse of notation, E^Q denotes expectation with respect to a product measure where Z-risk is evaluated as in the previous sections, and where interest rate risk is evaluated under the measure under Q. One can also write the market reserve in terms of the forward rates $f^\tau(t,y)$,

$$V^j(t,y) = E\left[\int_t^n e^{-\int_t^s f^\tau(t,y)d\tau} dB(s) \,\middle|\, Z(t) = j\right],$$

where expectation is taken over Z-risk only.

It should be emphasized that, in general, Y is a vector of processes and one speaks of an m-factor model if the dimension of Y is m. One example is to take Y to be one-dimensional and equate the short rate itself, i.e. $r(Y(t)) = Y(t) = r(t)$, which specifies a one-factor diffusion short rate model like e.g. the Vasicek model. Another example is to let the entries of Y be the forward rates corresponding to certain time horizons and then fill out the forward rate curve by some interpolation procedure. For other constructions we refer to Munk (2004).

For a factor diffusion model there exists a deterministic backward differential system for the reserve. This is a special case of the general Thiele differential equation derived in Steffensen (2000), and it reads

$$\begin{aligned}
\frac{\partial}{\partial t}V^{j}(t,y) &= V^{j}(t)\,r(y)-b^{j}(t)-\textstyle\sum_{k\neq j}\mu^{jk}(t)\,R^{jk}(t)\\
&\quad-\frac{\partial}{\partial y}V^{j}(t,y)\,(\alpha(y)-\beta(y)\,\eta(y))-\frac{1}{2}\frac{\partial^{2}}{\partial y^{2}}V^{j}(t,y)\,\beta(y)\,,\quad t\notin\mathcal{D}\,,\\
0 &= \Delta B^{j}(t)+V^{j}(t,y)-V^{j}(t-,y)\,,\ t\in\mathcal{D},\\
V^{j}(n,y) &= 0.
\end{aligned}$$

By Ito's formula we can now determine the dynamics of V^j,

$$\begin{aligned}
dV^{j}(t) &= \frac{\partial}{\partial t}V^{j}(t,Y(t))\,dt+\frac{\partial}{\partial y}V^{j}(t,Y(t))\,dY(t)\\
&\quad+\frac{1}{2}\frac{\partial^{2}}{\partial y^{2}}V^{j}(t,Y(t))\,\beta(Y(t))\,dt-\Delta B^{j}(t)\\
&= V^{j}(t)\,dG^{j}(t,Y(t))-dB^{j}(t)-\textstyle\sum_{k\neq j}\mu^{jk}(t)\,R^{jk}(t)\,dt,
\end{aligned}$$

where the capital gain process G^j is given by

$$dG^{j}(t,Y(t))=r(Y(t))\,dt-D^{j}(t,Y(t))\,\beta(Y(t))\,(\eta(Y(t))\,dt+dW(t))\,.$$

Here, the statewise factor duration $D^j(t,y)$ is given by

$$D^{j}(t,y)=-\frac{1}{V^{j}(t,y)}\frac{\partial}{\partial y}V^{j}(t,y)\,.$$

See also Munk (2004) for other results for factor diffusion models relevant for market valuation.

Acknowledgments

The author would like to thank Bo Søndergaard for pointing out the relevance of this study and for commenting on this note.

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