# FITTING RETURN PERIODS FOR LARGEST CLAIMS WITH A FRECHET COPULA: A CASE STUDY

Hürlimann Werner
Aon Re and IRMG (Switzerland) Ltd
Sternengasse 21, CH-4010 Basel
Tel.: +41 61 206 06 64
Fax.: +41 61 206 06 77
E-mail : werner.huerlimann@aon.ch
URL : www.geocities.com/hurlimann53

## Abstract

Based on a four-parameter large claims distribution, which combines an exponential distribution for the lower tail with a Pareto distribution for the upper tail, the fitting of long-term empirical return periods is examined. Two distinct methods are applied. First, the return period is fitted using the univariate survival function of the large claims mix over the considered lines of business. The second method uses the bivariate large claims probability that if a pair of claims from the two lines of business occurs, then one of its components exceeds a given largest claim amount. One notes a significant underestimation of the empirical return periods when using the univariate fit of the merged multi-line business. In contrast, the fit for the bivariate Fréchet exponential Pareto distribution is rather good. To round up the study, the impact of these distinct fitting methods on the expected value of excess-of-loss contracts is analyzed

## Key words

Pareto distribution, exponential distribution, Fréchet copula, return period, excess-of-loss reinsurance

## 1. Introduction.

In many lines of business, large claims are the driver of (re)insurance and are often responsible of more than 80% of the total amount of losses. In light of this background, it is very important to have a thorough look at large claims distributions. The present paper is devoted to fitting the return periods of largest claims for a two-line business, namely Property and Liability, whenever there is a long period of available historical observations.

Given claims statistics of a multi-line business, there are at least two ways to fit large claims distributions. First, it is possible to mix all claims without distinction in which line of business a claim actually occurred, and then fit an appropriate large claims distribution. A second method consists to fit a large claims distribution to each line of business separately and view these as margins of a multivariate large claims distribution. To represent these multivariate distributions, one uses copulas, which have found important applications in insurance and finance in recent years. A more detailed account of the content follows.

A method to fit univariate large claims distributions is presented in Section 2. While the extreme tail of a large claims distribution is adequately fitted by a Pareto distribution, its lower tail is sometimes rather poorly fitted. To remedy for this advantage, the lower tail is fitted using a two-parameter exponential distribution. Our experience has shown that this choice often fits better than a two-parameter gamma distribution. A fitting method comprising two steps is applied. In a first step, one determines the threshold and the Pareto index minimizing the chi-square statistic of the Pareto tail over some plausible set of threshold choices. In a second step, one determines the remaining parameters such that the chi-square value and the Cramér-von Mises $K$-statistic are sufficiently small. In Section 3, a bivariate large claims distribution is fitted using an extended Fréchet copula with univariate exponential Pareto large claims distributions. Given long-term empirical return periods of largest claims, we examine their fit based on two alternate large claims distributions. First, the return period is fitted using the univariate survival function of the large claims mix over the considered lines of business. The second method uses the bivariate large claims probability that if a pair of claims from the two lines of business occurs, then one of its components exceeds a given largest claim amount. Since one is only interested in largest claims exceeding a high threshold from a single line of business, the use of this survival function makes sense. One notes a significant underestimation of the empirical return periods when using the univariate fit of the merged Property and Liability business. In contrast, the fit for the bivariate Fréchet exponential Pareto distribution is rather good. Finally, the impact of these distinct fitting methods on the expected value of excess-of-loss contracts is analyzed in Section 5. In view of the diverging fit of the return periods, it is not surprising that the bivariate survival function yields in our illustration more than double higher values.

## 2. The univariate exponential Pareto large claims distribution.

It is well-known that the two-parameter Pareto distribution is an appropriate distribution often used to fit large claims distributions in (re)insurance. This has been a first choice in the practice of reinsurance for a long time (see e.g. Schmitter(1978), Schmitter and Bütikofer(1997), Doerr(1980), Schmutz and Doerr(1998)) and it is consistent with the theoretical results from Extreme Value Theory (e.g. Embrechts et al.(1997)).

Once the large claims distribution has been fitted in an adequate way, one often observes a rather poor fit in the lower tail of the distribution. To remedy for this disadvantage, it appears attractive to fit the lower tail using another simple two-parameter analytical distribution, for example a translated exponential distribution.

The considered combined four-parameter exponential Pareto distribution takes the form

$$F(x)=\begin{cases} 1-\exp\left(-\dfrac{x-\alpha}{\beta}\right), & \alpha\le x\le T, \\ 1-\exp\left(-\dfrac{T-\alpha}{\beta}\right)\cdot\left(\dfrac{x}{T}\right)^{-\gamma}, & x\ge T. \end{cases} \tag{2.1}$$

To fit this distribution to claims data, we proceed in two steps as follows. In a first step, one determines the threshold $T$ and the Pareto index $\gamma$ minimizing the chi-square statistic of the Pareto tail $\left(\frac{x}{T}\right)^{-\gamma}$, $x\ge T$, over some plausible set $[T_1,T_2]$ of threshold choices. In a second step, one determines the remaining parameters $\alpha, \beta$ such that the chi-square value and the Cramér-von Mises $K$-statistic are sufficiently small as follows.

Given a random sample $x_1, x_2, ..., x_n$ of claims data, let $y_i=\frac{i-1}{n-1}\in[0,1]$, $i=1,...,n$, be the corresponding percentile ranks. The fitted values of the distribution function are denoted by $\hat{y}_i=F(x_i)$, $i=1,...,n$. The chi-square value is defined by

$$\chi_2=\sum_{i=2}^{n}\frac{\left[y_i-y_{i-1}-(\hat{y}_i-\hat{y}_{i-1})\right]^2}{\hat{y}_i-\hat{y}_{i-1}}, \tag{2.2}$$

and the Cramér-von Mises $K$-statistic by

$$K=n\cdot\sum_{i=1}^{n}\frac{(\hat{y}_i-y_i)^2}{\hat{y}_i\cdot(1-\hat{y}_i)}. \tag{2.3}$$

The claims data of the case study used for illustration refers to the developed incurred claims of the Property and Liability business of an important industry over a recent reporting period of 4.75 years. Table A.1 in the Appendix lists this claims data. The second step of the fitting has been done using plausible smaller samples through elimination of immediate neighbor data points, which yield high contributions for the test statistics. The results and graphs of the statistical fit for the Property and Liability business, stand alone and merged, are found in the Tables A.2 to A.4.

## 3. The bivariate Fréchet exponential Pareto large claims distribution.

Though copulas have been introduced since Sklar(1959), their use in insurance and finance is quite recent. The method of copulas is a suitable form to represent bivariate distributions knowing their margins. The parameter of these copulas measures the degree of dependence between the margins. In case the widest possible range of dependence should be covered, one is especially interested in one-parameter families of copulas, which are able to model continuously a whole range of dependence between the lower Fréchet bound copula, the independent copula, and the upper Fréchet bound copula. Such families are called inclusive or comprehensive, and include the copulas by Frank(1979) and Clayton(1978). Another simple copula with this property is the following one, which has been used successfully in Hürlimann(2004).

Consider the following slight extension of the *Fréchet copula*, which is defined as follows. For $\theta \in [0,1]$ one has

$$C_\theta(u,v) = \begin{cases} [u+\theta(1-u)]\cdot v, & v \le u, \\ [v+\theta(1-v)]\cdot u, & v > u, \end{cases} \tag{3.1}$$

and for $\theta \in [-1,0]$ one has

$$C_\theta(u,v) = \begin{cases} (1+\theta)\cdot uv, & u+v<1, \\ uv+\theta\cdot(1-u)\cdot(1-v), & u+v \ge 1. \end{cases} \tag{3.2}$$

For $\theta \in [0,1]$ this copula is family B11 in Joe(1997), p. 148. It represents a mixture of perfect dependence and independence. If $X$ and $Y$ are uniform(0,1), $Y = X$ with probability $\theta$ and $Y$ is independent of $X$ with probability $1-\theta$, then $(X,Y)$ has the Fréchet copula. This distribution has been first considered by Konijn(1959) and motivated in Cohen(1960) along Cohen's kappa statistic (see Hutchinson and Lai(1990), Section 10.9). In previous work the defined copula has been called *linear Spearman copula*. The previously chosen nomenclature *linear* refers to the piecewise linear sections of this copula, and *Spearman* refers to the fact that the *grade correlation coefficient* $\rho_S$ coincides with the parameter $\theta$. This follows from the calculation

$$\rho_S = 12\cdot\int_0^1\int_0^1 [C_\theta(u,v)-uv]dudv = \theta\,, \tag{3.3}$$

where a proof of the integral representation is given in Nelsen(1991). The considered Fréchet copula, which leads to a so-called *Fréchet bivariate distribution*, has a singular component, which according to Joe should limit its field of applicability. Despite of this it has many interesting and important properties and is suitable for analytical computation. It has also been successfully fitted to financial data in Hürlimann(2004).

Let us return to the case study, and let $X$ represent the Property risk, $Y$ the Liability risk and $(X,Y)$ the bivariate Property and Liability risk. The statistical fits in Section 2 yield estimations of the distributions $F_X(x), F_Y(x)$ as well as of the distribution $F_Z(x)$, where $Z$ represents the merged Property and Liability risk arising by mixing the claims of both kinds that incur over time. There is an essential distinction between the distribution $F_Z(x)$ and the probability function $H(x) = P(X \le x, Y \le x)$. The first one represents the probability that if a claim from the Property or Liability business occurs, then it is less than $x$. The second one yields the probability that if a pair of claims from the Property and Liability business occurs, then each of its components is less than $x$. Based on long-term return period experience of largest claims, it will be shown in Section 4 that if one uses the distribution $F_Z(x)$, one observes a significant underestimation of the return periods, while if one uses $H(x)$ the obtained return periods fit rather well the observed ones.

The bivariate Fréchet exponential Pareto distribution, which is based on (3.1)-(3.2), leads to the following probability function

$$H(x) = \begin{cases} (1-\theta)F_X(x)F_Y(x) + \theta F^+(x), & \theta \ge 0, \\ (1+\theta)F_X(x)F_Y(x) - \theta F^-(x), & \theta \le 0, \end{cases} \tag{3.4}$$

where $F^{+}(x)$ and $F^{-}(x)$ are the distributions of comonotone and countermonotone copies of $X$ and $Y$, and are determined by the formulas

$$F^{+}(x) = \min\{F_X(x), F_Y(x)\}, \quad (3.5)$$
$$F^{-}(x) = \max\{F_X(x) + F_Y(x) - 1, 0\}. \quad (3.6)$$

## 4. Fitting long-term return periods of largest claims.

While the claims data of our case study is restricted to a recent and relatively short reporting period, there is for the considered industry a long and over one-hundred years period of experience for the occurred largest claims. Developed to actual prices, the return periods of some of these claims are listed in Table 4.1.

**Table 4.1 :** empirical return periods of the largest claims

| Large claim limit (in Mio.) | Period of observation | Observed return period |
|---|---|---|
| 5 | 1982-2002 | 1.31 |
| 10 | 1924-2002 | 3.59 |
| 15 | 1891-2002 | 6.58 |
| 25 | 1891-2002 | 16 |

Given a large claims survival function $S(x)$ and the known observed one-year frequency of claims above a threshold $T$, say $\lambda_T$, the theoretical return period of a large claim $x > T$ is given by

$$r(x) = \frac{1}{S(x) \cdot \lambda_T} \quad (4.1)$$

According to Table A.1 the observed one-year frequency for the threshold $T = 1'064'000$ is $\lambda_T = 43/4.75 = 9.05$. In Table 4.1 the return periods for the fitted large claims survival functions $S_Z(x) = 1 - F_Z(x)$ and $S(x) = 1 - H(x)$ are compared. Again, there is an essential distinction in interpretation. The first one yields the probability that if a claim from the Property or Liability business occurs, then it exceeds *x*. The second one represents the probability that if a pair of claims from the Property and Liability business occurs, then one of its components exceeds *x*. Since one is only interested in largest claims exceeding a high threshold from a single line of business, the use of the survival function $S(x)$ makes sense. The dependence parameter of the bivariate Fréchet exponential Pareto distribution is chosen such that it minimizes the chi-square value for the observed return periods, namely

$$\chi_2 = \sum_{k=1}^{4} \frac{(\hat{r}(x_k) - r(x_k))^2}{r(x_k)}, \quad (4.2)$$

where $(x_1, x_2, x_3, x_4) = (5,10,15,25)$ and the observed return period $\hat{r}(x_k)$ is listed in Table 4.1. For the minimizing choice $\theta = 0$ (independent Property and Liability risks), one has $\chi_2 = 0.1$ while for the univariate survival function $S_Z(x)$ one has $\chi_2 = 21.07$.

**<u>Table 4.1</u>:** comparison of return periods

| Large claim limit | Return periods | | |
|---|---|---|---|
| (in Mio.) | Univariate fit | Bivariate fit | Observed |
| 5 | 2.59 | 1.17 | 1.31 |
| 10 | 8.18 | 3.75 | 3.59 |
| 15 | 16.04 | 7.27 | 6.58 |
| 25 | 37.45 | 16.39 | 16 |

One notes a significant underestimation of the empirical return periods when using the univariate fit of the merged Property and Liability business. In contrast, the fit for the bivariate Fréchet exponential Pareto distribution is rather good. It is worthwhile to mention that the bivariate survival function $S(x)$ is more dangerous than the univariate survival function $S_Z(x)$ in the stochastic order, that is $S_Z(x) \le S(x)$ for all $x$.

## 5. <u>The impact on the XL reinsurance of largest claims</u>.

In the framework of the classical *collective model* of risk theory, the *aggregate claims* of a portfolio of insurance risks are described by the random variable

$$S = \sum_{i=1}^{N} X_i \,, \tag{5.1}$$

where the *claim sizes* $X_i$ are independent and identically distributed and independent from the random *claim number* $N$.

An *excess-of-loss* or *XL-reinsurance* treaty with *deductible* $d$ on a portfolio of risks covers for each claim $X_i$ the *excess claim size* $(X_i - d)_+$, $i = 1,\ldots,N$. In this setting, the *retained aggregate claims* of the cedant are described by the random variable

$$S_c = \sum_{i=1}^{N} \min(X_i, d), \tag{5.2}$$

and the *reinsured aggregate claims* are given by

$$S_r = \sum_{i=1}^{N} (X_i - d)_+ \,. \tag{5.3}$$

Since the independence assumptions are preserved under the transformations of the claim sizes, both (5.2) and (5.3) are again collective models. However, a reinsurer does not know the number and the size of the original claims below the deductible $d$. Therefore, the collective model (5.3) is not appropriate to forecast the loss of the reinsurer. Fortunately, it is possible to construct a collective model for the reinsurer on the basis of the collective model for the original claims such that the model contains only random variables which are observable for the reinsurer. This collective model for the reinsurer is presented in Hess(2003) and the related literature in Hess et al.(1995), Franke and Macht(1995), Mack(1997) and Schmidt(1996/2002).

Suppose that the reinsurer relies on the claims data provided by the insurers, where only the claims above a limit $OP$, called *observation point*, are reported. This means that only forecasts for XL-reinsurance with a deductible $d \ge OP$ can be made. Denote by $X$ a random variable distributed as $X_i$ for all $i = 1,\ldots,N$. It is assumed that $P(X > OP) > 0$, that is the probability that a claim exceeds $OP$ is strictly positive. The aggregate claims of the reinsurer, which contains all claims exceeding $OP$, is described by the collective model of risk theory

$$S_r^{OP} = \sum_{i=1}^{N^{op}} X_i^{OP} \, . \tag{5.4}$$

In this expression $N^{OP}$ counts the number of claims above $OP$ and is given by

$$N^{OP} = \sum_{i=1}^{N} B_i \, , \tag{5.5}$$

where the $B_i$'s are independent and identically distributed Bernoulli random variables, which are independent from $N$, such that $P(B_i = 1) = P(X > OP)$. The claim sizes $X_i^{OP}$ are independent and identically distributed, and independent from $N$, and each is distributed like $X^{OP}$ with distribution $P(X^{OP} \le x) = P(X \le x | X > OP), \quad x > OP$. This is the basic collective model of risk theory used in XL-reinsurance.

We assume that the *aggregate claims* of a portfolio of insurance risks are described by the random variable

$$S = \sum_{i=1}^{N_T} X_i \, , \tag{5.6}$$

where the *claim sizes* $X_i$ are independent and identically distributed and independent from the random *claim number* $N_T$. The variable $N_T$ counts the number of claims above a fixed observation point $T$ and is assumed to be Poisson distributed with parameter $\lambda_T = E[N_T]$. Denote by $X$ a random variable distributed as $X_i$.

An *excess-of-loss* (XL) *reinsurance* program $C \; xs \; D$ covers the amount of each claim that exceeds the deductible $D$ up to the maximum cover of $C$. Based on the survival function $S_Z(x)$, the expected value of the aggregate claims of the XL reinsurance cover is given by $\lambda_T \cdot \{m_Z(D) - m_Z(C+D)\}$ with

$$m_Z(x) = \int_x^\infty S_Z(z)dz = \exp\left\{-\frac{T-\alpha}{\beta}\right\} \cdot \frac{T}{\gamma-1} \cdot \left(\frac{x}{T}\right)^{-(\gamma-1)} \, . \tag{5.7}$$

Similarly, for the bivariate survival function $S(x)$, the expected value of the aggregate claims of the XL layer is given by $\lambda_T \cdot \{m(D) - m(C+D)\}$ with

$$m(x) = \int_x^\infty S(z)dz . \tag{5.8}$$

In the special case $\theta = 0$ of independent risks, as in our case study, one obtains the analytical formula

$$\begin{aligned} m(x) = \exp\left\{-\frac{T_X - \alpha_X}{\beta_X}\right\} \cdot \frac{T_X}{\gamma_X - 1} \cdot \left(\frac{x}{T_X}\right)^{-(\gamma_X - 1)} + \exp\left\{-\frac{T_Y - \alpha_Y}{\beta_Y}\right\} \cdot \frac{T_Y}{\gamma_Y - 1} \cdot \left(\frac{x}{T_Y}\right)^{-(\gamma_Y - 1)} \\ - \exp\left\{-\frac{T_X - \alpha_X}{\beta_X} - \frac{T_Y - \alpha_Y}{\beta_Y}\right\} \cdot \left(\frac{T}{T_X}\right)^{-\gamma_X} \cdot \left(\frac{T}{T_Y}\right)^{-\gamma_Y} \cdot \frac{T}{\gamma_X + \gamma_Y - 1} \cdot \left(\frac{x}{T}\right)^{-(\gamma_X + \gamma_Y - 1)} , \end{aligned} \tag{5.9}$$

where the parameters refer to exponential Pareto distributions of the risks $X, Y, Z$, and take the values shown in the Tables A.2-A.4 of the Appendix. With $\lambda_T = 43/4.75 = 9.05$ for $T = 1'064'000$, one obtains the following expected values for the XL reinsurance of high layers. In view of the return periods obtained in Section 4, it is not surprising that the bivariate survival function yields more than double higher values.

**<u>Table 5.1</u>:** Comparison of expected aggregate claims for high XL reinsurance layers

| XL layer (in Mio.) | Expected aggregate claims of XL layer | |
|---|---|---|
| | Univariate fit | Bivariate fit |
| 5 xs 25 | 114'675 | 264'265 |
| 10 xs 25 | 201'427 | 467'741 |
| 15 xs 25 | 269'762 | 630'721 |
| 20 xs 25 | 325'237 | 765'127 |
| 25 xs 25 | 371'335 | 878'483 |
| 30 xs 25 | 410'363 | 975'800 |
| 40 xs 25 | 473'117 | 1'135'275 |
| 50 xs 25 | 521'636 | 1'261'537 |

**Appendix:** Fitting exponential Pareto large claims distributions

**Table A.1:** Property and Liability large claims data

| Property | | Liability |
|---|---|---|
| 10'050'000 | 985'000 | 5'492'655 |
| 7'682'827 | 985'000 | 3'646'519 |
| 5'037'264 | 982'000 | 2'500'000 |
| 4'036'099 | 961'494 | 1'922'760 |
| 3'033'243 | 900'000 | 1'912'228 |
| 2'890'985 | 835'000 | 1'897'058 |
| 2'500'000 | 821'829 | 1'778'931 |
| 2'500'000 | 800'000 | 1'624'418 |
| 2'182'000 | 782'000 | 1'562'243 |
| 2'000'000 | 776'045 | 1'544'041 |
| 1'998'000 | 757'906 | 1'442'070 |
| 1'985'000 | 756'005 | 1'396'261 |
| 1'985'000 | 742'000 | 1'265'392 |
| 1'983'871 | 655'661 | 1'231'404 |
| 1'982'000 | 640'000 | 1'153'656 |
| 1'735'000 | 600'000 | 1'102'500 |
| 1'495'510 | 597'801 | 1'080'450 |
| 1'485'000 | 585'000 | 980'000 |
| 1'350'000 | 580'000 | 913'352 |
| 1'339'448 | 575'000 | 901'294 |
| 1'285'000 | 571'394 | 781'121 |
| 1'246'105 | 549'092 | 666'318 |
| 1'222'713 | 531'904 | 659'994 |
| 1'185'000 | 526'386 | 649'114 |
| 1'155'834 | | 614'195 |
| 1'123'460 | | 564'883 |
| 1'050'000 | | 564'811 |
| 989'735 | | 551'415 |
| 986'010 | | 527'158 |

**Table A.2:** Exponential Pareto fit of the Property risk

| Claims data | Percentile Rank | Exponential - Pareto Fit | Chi-Square Test | Cramér - von Mise K |
|---|---|---|---|---|
| 526'386 | 0.00% | 0.78% | 0.00000 | 0.00782 |
| 549'092 | 2.94% | 3.49% | 0.00020 | 0.00088 |
| 571'394 | 5.88% | 6.08% | 0.00048 | 0.00007 |
| 580'000 | 8.82% | 7.06% | 0.03920 | 0.00477 |
| 597'801 | 11.76% | 9.05% | 0.00448 | 0.00894 |
| 640'000 | 14.71% | 13.61% | 0.00576 | 0.00101 |
| 655'661 | 17.65% | 15.25% | 0.01045 | 0.00445 |
| 742'000 | 20.59% | 23.72% | 0.03609 | 0.00541 |
| 756'005 | 23.53% | 25.01% | 0.02106 | 0.00117 |
| 776'045 | 26.47% | 26.82% | 0.00706 | 0.00006 |
| 800'000 | 29.41% | 28.93% | 0.00330 | 0.00011 |
| 821'829 | 32.35% | 30.79% | 0.00618 | 0.00114 |

**Table A.2:** Exponential Pareto fit of the Property risk - continued

| Claims data | Percentile Rank | Exponential - Pareto Fit | Chi-Square Test | Cramér - von Mise K |
|---|---|---|---|---|
| 835'000 | 35.29% | 31.90% | 0.03065 | 0.00531 |
| 900'000 | 38.24% | 37.09% | 0.00974 | 0.00057 |
| 961'494 | 41.18% | 41.63% | 0.00566 | 0.00009 |
| 982'000 | 44.12% | 43.07% | 0.01560 | 0.00044 |
| 1'050'000 | 47.06% | 47.86% | 0.00714 | 0.00026 |
| 1'123'460 | 50.00% | 52.68% | 0.00732 | 0.00289 |
| 1'155'834 | 52.94% | 54.57% | 0.00586 | 0.00107 |
| 1'185'000 | 55.88% | 56.17% | 0.01136 | 0.00003 |
| 1'222'713 | 58.82% | 58.09% | 0.00535 | 0.00022 |
| 1'246'105 | 61.76% | 59.22% | 0.02940 | 0.00269 |
| 1'285'000 | 64.71% | 60.98% | 0.00795 | 0.00585 |
| 1'339'448 | 67.65% | 63.23% | 0.00209 | 0.00839 |
| 1'485'000 | 70.59% | 68.29% | 0.00885 | 0.00245 |
| 1'735'000 | 73.53% | 74.63% | 0.01824 | 0.00064 |
| 1'982'000 | 76.47% | 79.04% | 0.00488 | 0.00398 |
| 2'182'000 | 79.41% | 81.74% | 0.00022 | 0.00362 |
| 2'500'000 | 82.35% | 84.97% | 0.00027 | 0.00538 |
| 2'890'985 | 85.29% | 87.80% | 0.00005 | 0.00587 |
| 3'033'243 | 88.24% | 88.61% | 0.05582 | 0.00014 |
| 4'036'099 | 91.18% | 92.44% | 0.00205 | 0.00229 |
| 5'037'264 | 94.12% | 94.50% | 0.00379 | 0.00028 |
| 7'682'827 | 97.06% | 97.00% | 0.00078 | 0.00001 |
| 10'050'000 | 100.00% | 97.96% | 0.04089 | 0.02086 |
| 15'000'000 | 100.00% | 98.85% | 0.00893 | 0.01164 |
| 25'000'000 | 100.00% | 99.45% | 0.00597 | 0.00556 |
| **Chi-square value / K-statistic :** | | | **0.423** | **4.68** |

**Figure A.1:** Exponential Pareto fit of the Property risk

| Scale $\beta$ | Location $\alpha$ | Observation Point T | Pareto index $\gamma$ |
|---|---|---|---|
| **820'000** | **520'000** | **1'020'000** | **1.43414** |

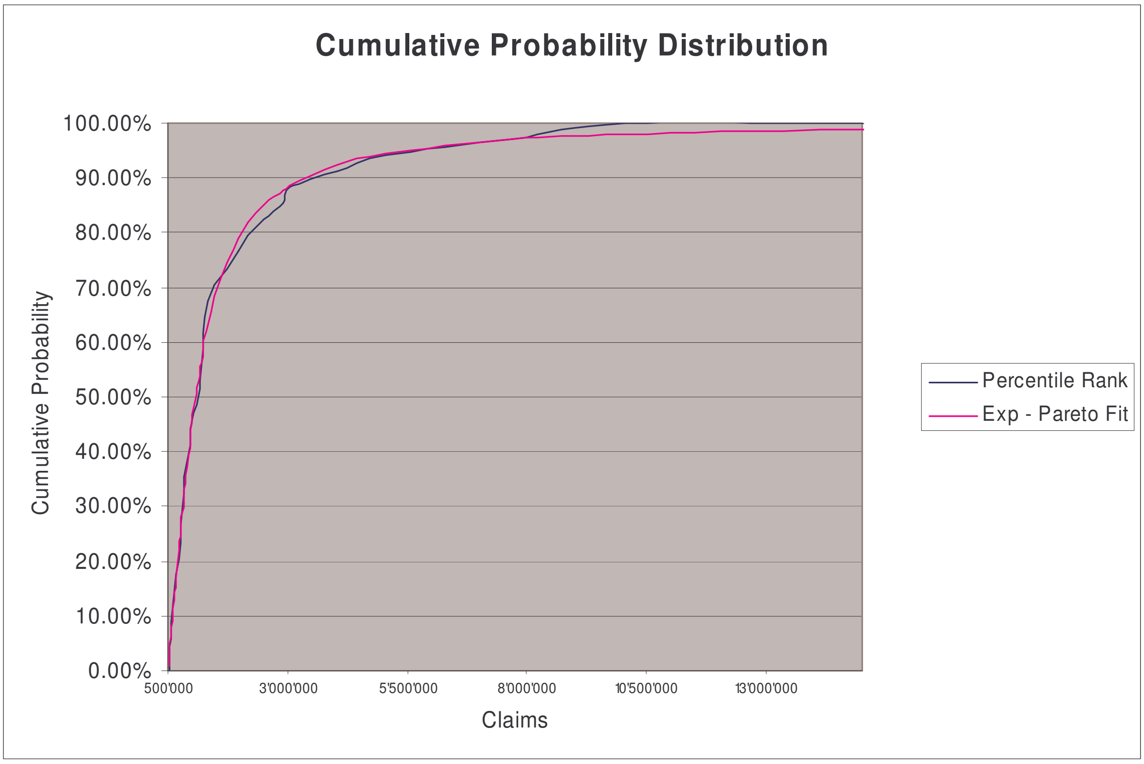



**Table A.3:** Exponential Pareto fit of the Liability risk

| Claims data | Percentile Rank | Exponential - Pareto Fit | Chi-Square Test | Cramér - von Mise K |
|---|---|---|---|---|
| 527'158 | 0.00% | 1.21% | 0.00000 | 0.01223 |
| 551'415 | 5.00% | 3.58% | 0.02927 | 0.00588 |
| 614'195 | 10.00% | 9.44% | 0.00128 | 0.00036 |
| 649'114 | 15.00% | 12.55% | 0.01152 | 0.00546 |
| 666'318 | 20.00% | 14.04% | 0.08251 | 0.02940 |
| 781'121 | 25.00% | 23.37% | 0.02004 | 0.00149 |
| 901'294 | 30.00% | 32.04% | 0.01559 | 0.00192 |
| 980'000 | 35.00% | 37.19% | 0.00004 | 0.00205 |
| 1'080'450 | 40.00% | 43.19% | 0.00168 | 0.00415 |
| 1'153'656 | 45.00% | 47.20% | 0.00244 | 0.00194 |
| 1'231'404 | 50.00% | 52.37% | 0.00005 | 0.00224 |
| 1'265'392 | 55.00% | 55.12% | 0.01820 | 0.00001 |
| 1'396'261 | 60.00% | 63.83% | 0.01578 | 0.00636 |
| 1'442'070 | 65.00% | 66.30% | 0.02590 | 0.00076 |
| 1'544'041 | 70.00% | 70.99% | 0.00021 | 0.00047 |
| 1'624'418 | 75.00% | 74.04% | 0.01241 | 0.00048 |
| 1'778'931 | 80.00% | 78.73% | 0.00021 | 0.00097 |
| 1'897'058 | 85.00% | 81.52% | 0.01738 | 0.00803 |
| 2'500'000 | 90.00% | 89.91% | 0.01367 | 0.00001 |
| 3'646'519 | 95.00% | 95.59% | 0.00081 | 0.00082 |
| 5'492'655 | 100.00% | 98.20% | 0.02177 | 0.01831 |
| 15'000'000 | 100.00% | 99.80% | 0.01599 | 0.00199 |
| 25'000'000 | 100.00% | 99.94% | 0.00134 | 0.00065 |
| **Chi-square value / K-statistic :** | | | **0.308** | **2.44** |

**Figure A.2:** Exponential Pareto fit of the Liability risk

| Scale $\beta$ | Location $\alpha$ | Observation Point T | Pareto index $\gamma$ |
|---|---|---|---|
| 1'000'000 | 515'000 | 1'200'000 | 2.19147 |

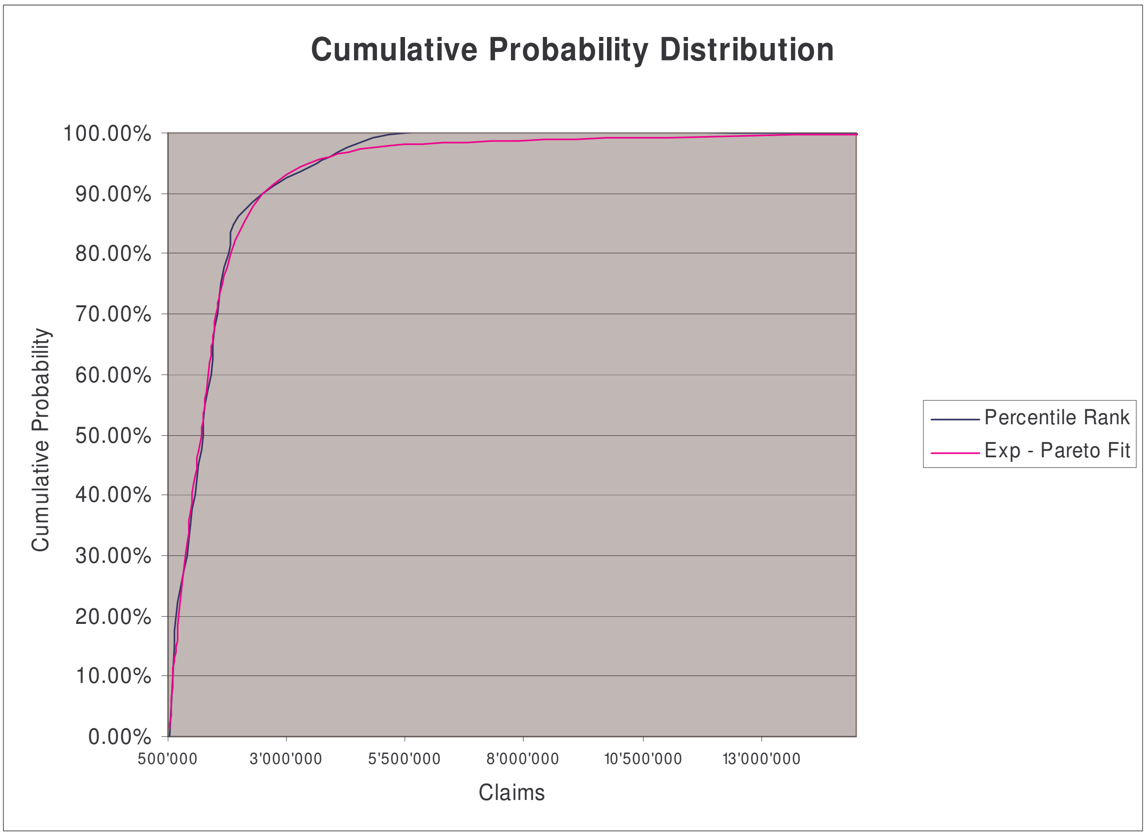



**Table A.4:** Exponential Pareto fit of the merged Property and Liability risk

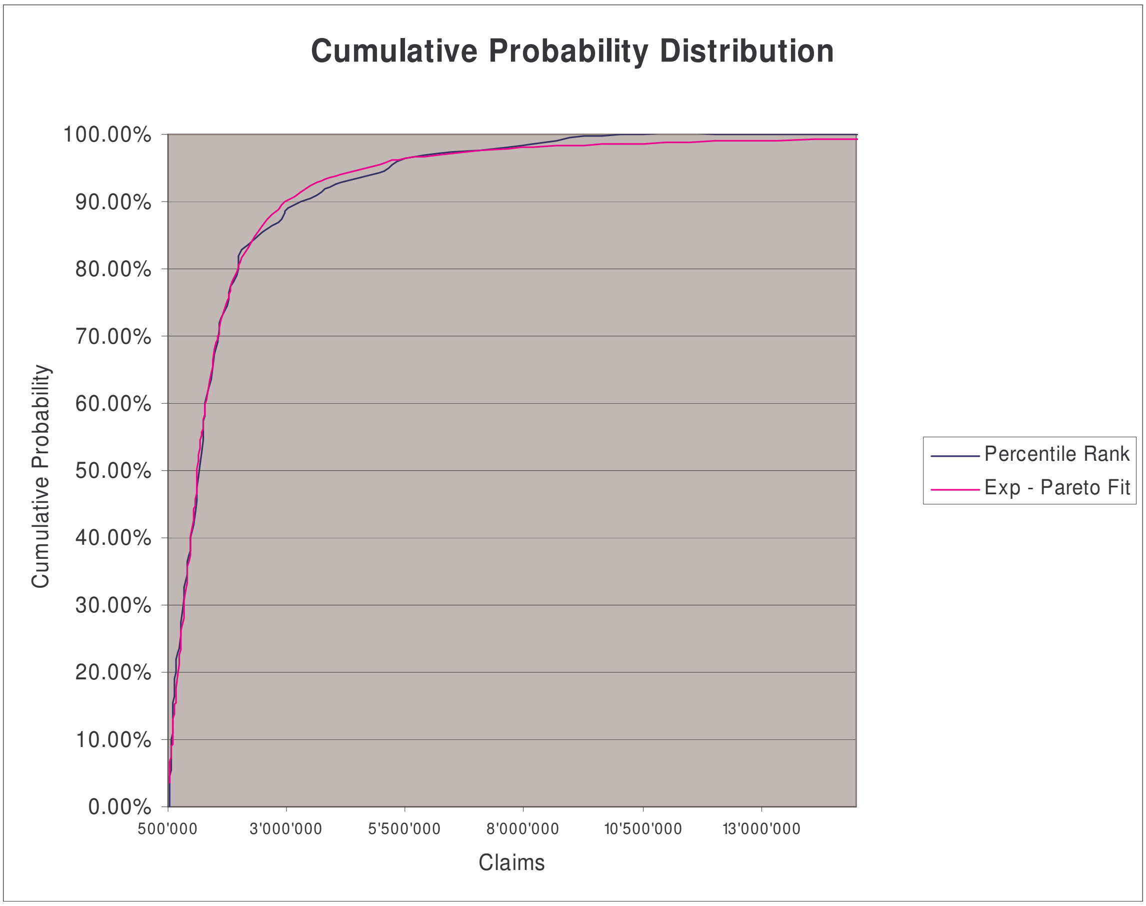
| Claims data | Percentile Rank | Exponential - Pareto Fit | Chi-Square Test | Cramér - von Mise K |
|---|---|---|---|---|
| 526'386 | 0.00% | 3.64% | 0.00000 | 0.03783 |
| 531'904 | 1.82% | 4.19% | 0.03015 | 0.01398 |
| 549'092 | 3.64% | 5.85% | 0.00014 | 0.00891 |
| 564'811 | 5.45% | 7.35% | 0.00068 | 0.00527 |
| 571'394 | 7.27% | 7.97% | 0.02314 | 0.00066 |
| 580'000 | 9.09% | 8.77% | 0.01277 | 0.00012 |
| 585'000 | 10.91% | 9.24% | 0.03949 | 0.00333 |
| 597'801 | 12.73% | 10.42% | 0.00348 | 0.00572 |
| 614'195 | 14.55% | 11.90% | 0.00074 | 0.00666 |
| 640'000 | 16.36% | 14.19% | 0.00097 | 0.00387 |
| 649'114 | 18.18% | 14.99% | 0.01320 | 0.00801 |
| 655'661 | 20.00% | 15.55% | 0.02769 | 0.01506 |
| 666'318 | 21.82% | 16.47% | 0.00896 | 0.02083 |
| 742'000 | 23.64% | 22.67% | 0.03104 | 0.00053 |
| 756'005 | 25.45% | 23.77% | 0.00474 | 0.00156 |
| 776'045 | 27.27% | 25.31% | 0.00049 | 0.00203 |

**Table A.4:** Exponential Pareto fit of the merged Property and Liability risk - continued

| Claims data | Percentile Rank | Exponential - Pareto Fit | Chi-Square Test | Cramér - von Mise K |
|---|---|---|---|---|
| 800'000 | 29.09% | 27.12% | 0.00000 | 0.00197 |
| 821'829 | 30.91% | 28.72% | 0.00028 | 0.00233 |
| 835'000 | 32.73% | 29.67% | 0.00789 | 0.00446 |
| 900'000 | 34.55% | 34.19% | 0.01609 | 0.00006 |
| 913'352 | 36.36% | 35.08% | 0.00966 | 0.00073 |
| 961'494 | 38.18% | 38.19% | 0.00538 | 0.00000 |
| 980'000 | 40.00% | 39.35% | 0.00379 | 0.00018 |
| 1'050'000 | 41.82% | 43.53% | 0.01336 | 0.00119 |
| 1'080'450 | 43.64% | 45.73% | 0.00067 | 0.00176 |
| 1'102'500 | 45.45% | 47.52% | 0.00000 | 0.00171 |
| 1'123'460 | 47.27% | 49.13% | 0.00025 | 0.00139 |
| 1'153'656 | 49.09% | 51.33% | 0.00063 | 0.00200 |
| 1'185'000 | 50.91% | 53.44% | 0.00043 | 0.00258 |
| 1'222'713 | 52.73% | 55.80% | 0.00124 | 0.00384 |
| 1'231'404 | 54.55% | 56.32% | 0.03280 | 0.00128 |
| 1'246'105 | 56.36% | 57.17% | 0.01095 | 0.00027 |
| 1'265'392 | 58.18% | 58.25% | 0.00508 | 0.00000 |
| 1'285'000 | 60.00% | 59.30% | 0.00558 | 0.00020 |
| 1'339'448 | 61.82% | 62.01% | 0.00293 | 0.00002 |
| 1'396'261 | 63.64% | 64.54% | 0.00201 | 0.00036 |
| 1'442'070 | 65.45% | 66.39% | 0.00001 | 0.00039 |
| 1'485'000 | 67.27% | 67.99% | 0.00031 | 0.00024 |
| 1'544'041 | 69.09% | 70.00% | 0.00018 | 0.00039 |
| 1'562'243 | 70.91% | 70.57% | 0.02661 | 0.00005 |
| 1'624'418 | 72.73% | 72.42% | 0.00000 | 0.00005 |
| 1'735'000 | 74.55% | 75.28% | 0.00377 | 0.00029 |
| 1'778'931 | 76.36% | 76.28% | 0.00657 | 0.00000 |
| 1'897'058 | 78.18% | 78.68% | 0.00141 | 0.00015 |
| 1'982'000 | 80.00% | 80.18% | 0.00070 | 0.00002 |
| 1'998'000 | 81.82% | 80.44% | 0.09205 | 0.00121 |
| 2'182'000 | 83.64% | 83.10% | 0.00267 | 0.00020 |
| 2'500'000 | 85.45% | 86.52% | 0.00747 | 0.00097 |
| 2'890'985 | 87.27% | 89.41% | 0.00397 | 0.00481 |
| 3'033'243 | 89.09% | 90.22% | 0.01247 | 0.00144 |
| 3'646'519 | 90.91% | 92.79% | 0.00223 | 0.00532 |
| 4'036'099 | 92.73% | 93.91% | 0.00440 | 0.00246 |
| 5'037'264 | 94.55% | 95.79% | 0.00002 | 0.00381 |
| 5'492'655 | 96.36% | 96.35% | 0.02789 | 0.00000 |
| 7'682'827 | 98.18% | 97.91% | 0.00043 | 0.00036 |
| 10'050'000 | 100.00% | 98.66% | 0.01510 | 0.01357 |
| 15'000'000 | 100.00% | 99.31% | 0.00650 | 0.00694 |
| 25'000'000 | 100.00% | 99.71% | 0.00394 | 0.00296 |
| **Chi-square value / K-statistic :** | | | **0.535** | **11.97** |

**Figure A.3:** Exponential Pareto fit of the merged Property and Liability risk

| Scale $\beta$ | Location $\alpha$ | Observation Point T | Pareto index $\gamma$ |
|---|---|---|---|
| 980'000 | 490'000 | 1'064'000 | 1.65999 |

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