Insurer risk management in the presence of frictional costs [⋆]

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Abstract

The paper introduces modelling of the financial distress costs in an insurance company as a corporation with risky debt, and investigates the existence of risk management incentives in insurance business in the presence of financial distress costs.

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1 Introduction

One of the main incentives to manage corporate risk assumed by a corporation, is that variability in financial outcomes is costly when the corporation operates in an environment with frictional costs. The risk management strategies depend on the nature of the frictional costs. There are essentially three sources of frictional capital costs: *tax asymmetry*, *costs of financial distress* and *agency costs*. In the insurance industry, where the market is regulated, an insurer, which is effectively leveraged by risky insurance debt, experiences an additional two costs of corporate risk: *cost of double taxation* and *cost of regulatory restrictions*. This paper investigates the strategies of an insurance firm's risk management in the maximization of shareholders' value under the presence of financial distress costs. Briefly financial distress can be defined as a low net-worth (surplus) state of the insurance company in which the insurer incurs additional deadweight losses. The notion that financial distress is a different state from legal *insolvency* has been introduced in the financial literature (e.g. see paper by Diamond (1991 [4]), where the model of an illiquid but solvent firm is considered, and the paper by Froot et al. (1993 [7]), where the model of a firm with low cash-flow is considered; also see the paper by Briys and de Varenne (1997 [2]) and Jarrow and Purnanandam (2004 [9]), and references therein). Insurance (underwriting) risk increases the probability that an insurer will experience financial distress. Financial distress can be costly due to both direct costs, such as

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legal fees (third party costs), lost value from distressed sales ("fire sale" losses), and indirect costs, mainly loss of reputation and franchise value. Some empirical studies (e.g. Opler and Titman (1994 [14]), Andrade and Kaplan (1998 [1])) of financial companies have revealed that financial distress results in costs of around $10 - 20\%$ of market value of assets. These costs are likely to be higher in the insurance industry due to the credit-sensitive nature of policyholders. Insurers with a substantial value of insurance outstanding claim liabilities will be exposed to a higher risk of being financially distressed. Therefore, highly leveraged insurers have an incentive to control their risk level. This can be achieved by either holding a greater amount of risk capital (i.e. excess of minimum amount required by regulator) or by the use of risk transfer, such as traditional reinsurance or alternative risk transfer (ART) products (Culp (2002 [3])). However, holding an extra amount of capital in an insurance company is costly due to other frictions and market imperfections, such as agency costs, costs arising from adverse selection and moral hazard, and regulatory costs (see Merton and Perold (1998 [12])). Because of these and other capital costs, insurers may avoid holding more than the minimal amount of risk capital required by a regulator to reduce the probability of financial distress event. Hence, risk transfer can be preferable. An additional benefit of efficiently controlling risk via risk transfer (reinsurance) is that it reduces the level of unnecessary volatility in the profit and loss statement. Reducing volatility helps insurers to create sustainable shareholders value. This paper continues the research on risk management incentives in a single period model with frictional costs, started in Krvavych and Sherris (2006 [11]), and considers risk management in two different single period models of the insurance underwriting process with financial distress (FD) costs. In the first model we consider the insurer's solvency ratio, and in the second model we consider risk based capital. In the first model we model the dynamics of the company's liabilities and its solvency ratio with geometric Brownian motions. Assuming that all liabilities are paid at the end of the period, the insurer becomes financially distressed at the end of the period if the solvency ratio falls below a pre-specified FD barrier. Insolvency occurs if the solvency ratio falls below one. The objective function in this model is to maximize the expected shareholders' terminal payoff (valuable shareholders' claim on the terminal value of company's assets net of liabilities and FD costs). In the second model we model the company's surplus (net-worth) with a geometric Brownian motion, and consider the underwriting process over the period of time between two consecutive audits. At the beginning of the period the company is solvent, i.e. it satisfies minimum solvency requirements (minimal risk capital capacity or regulatory surplus) imposed by a regulator. The firm is financially distressed if the net-worth of the company falls below a threshold (barrier of financial distress, usually lower than required minimal level of risk capital) during the period of time between audits. Insolvency[2] occurs at the end of the period (on an auditing date) if the terminal net-worth of the company is below the required minimal level of risk capital. We consider the terminal value of excess of the required regulatory surplus, and assume that a solvent company retains a part of this excess of regulatory surplus (i.e. retained ratio times excess of surplus), and the rest is paid to shareholders as dividends. We maximize the expected terminal value of the company's surplus net of regulatory surplus and FD costs, which depends on the terminal value of the company's surplus as well as on the path of the company's surplus during the period. In this paper we consider four different forms of financial distress costs. These are

1) proportional to the terminal value of the insurer's assets;
2) proportional to the terminal value of the insurer's surplus (assets minus liabilities);

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[2] We use term *insolvency* to define the economic state in which an insurance company is *undercapitalized* from a regulator's point of view and cannot continue its underwriting process without reorganizing its capital structure.

3) constant;
4) reduced upside potential of the terminal value of insurer's surplus.

In the case of FD of the fourth form we assume that in the event of financial distress, the company's operations are adversely affected in such way that it is unable to realize the full upside potential of its surplus. Such representation of FD costs is consistent with the idea that a financial distressed insurance company is unable to capitalize on its real options and suffers reduced growth as a result. Although this form of FD costs is rather theoretical (technical) than practical, the use of it has advantages in obtaining an analytical (exact) form of the optimal value of the company's aggregate investment-underwriting risk as a function of the financial distress barrier, the deadweight losses caused by financial distress, the minimal value of surplus required by a regulator and the length of the period. At the same time we can detect incentives to control the company's aggregate investment-underwriting risk in the models of the maximization of shareholders value under the presence of FD costs of other three forms using numerical calculations only. In the first model we consider the deadweight losses caused by financial distress that are proportionate to the terminal value of the company's assets. We investigate the second model of the maximization of shareholders' value under the presence of of the other forms of FD costs.

2 Models setup and definitions: SR and RBC solvency models

We define two solvency models: the solvency ratio (SR) model and the risk based capital (RBC) model.

2.1 The SR model

Similarly to Sherris and van der Hoek (2006 [17]) we consider an insurer's underwriting process in the period of time from 0 to T and model the risk-neutral dynamics of insurer's liabilities and solvency ratio with geometric Brownian motions. Denote the value of company's liabilities at time t by L_t for $t \in [0, T]$. Assume that the $\mathbb{Q}$-risk-neutral dynamics of L_t are

$$dL_t = \mu_L L_t \, dt + \sigma_L L_t \, dW_t^L, \quad t \in [0, T], \tag{1}$$

and that there are no claim payments made other than at the end of the period, i.e. L_T. If in addition L can be replicated by traded assets then $\mu_L = r$, the risk free rate. We know that the insurance policies are contingent claims on the value of the liabilities with payoff that depends on the insurer solvency and assume that in the SR model the solvency is measured by solvency ratio

$$\Lambda_t = \frac{A_t}{L_t},$$

where A_t denotes the value of the company's assets at time $t \in [0, T]$. We assume that the $\mathbb{Q}$-risk-neutral dynamics of Λ_t are

$$d\Lambda_t = \mu_\Lambda \Lambda_t \, dt + \sigma_\Lambda \Lambda_t \, dW_t^\Lambda, \quad t \in [0, T], \tag{2}$$

with $\Lambda_0 = \frac{A_0}{L_0} > 1$, i.e. the insurer is solvent at the beginning of the period. As in Sherris and van der Hoek (2006 [17]) it is assumed that the parameters μ_Λ and σ_Λ are known and given (estimated from data). We define three different economic states of the insurance company:

- *financially distressed and solvent*;
- *financially healthy*;
- *insolvent.*

The insurer becomes financially distressed if the terminal value of the solvency ratio Λ_T falls below a pre-specified threshold (FD barrier) b (i.e. $\Lambda_T \leq b$). In the state of FD the company experiences deadweight losses proportionate to the terminal value of assets A_T with proportionate coefficient $1 - w,\ w \in [0, 1)$. Being financially distressed the insurer is solvent if the terminal value of assets net of FD costs exceeds the terminal value of liabilities, i.e. when $w\, A_T > L_T$ or $\Lambda_T > \frac{1}{w}$. If $w\, b < 1$ (or $b < \frac{1}{w}$) then financial distress implies immediate insolvency, and thus the two states financial distress and insolvency coincide. Another consideration is that the insurer holds risk capital as a part of its total assets to back up its liabilities and hence the solvency ratio of financially healthy insurer is considerably higher than 1. If the solvency ratio is quite close to 1, then more likely the insurer experiences shortage in the necessary risk capital, i.e. it is financially distressed. Therefore, we assume that $b > \frac{1}{w}$ which also ensures that there is a chance of recovering from the FD state. It is obvious that if the insurer is financially healthy (i.e. $\Lambda_T > b$) then it is solvent, since $\Lambda_T > b > \frac{1}{w} > 1$. Within the SR model the expected present value of shareholders' terminal payoff is equal to

$$\begin{aligned}
V_0^{SR} &= \mathbb{E}_{\mathbb{Q}} \left[\mathrm{e}^{-rT} \left\{ (w\, A_T - L_T)^+ \, \mathbf{1}_{\{\Lambda_T \leq b\}} + (A_T - L_T) \, \mathbf{1}_{\{\Lambda_T > b\}} \right\} \right] \\
&= \mathbb{E}_{\mathbb{Q}} \left[\mathrm{e}^{-rT} L_T \left\{ w \left(\Lambda_T - \frac{1}{w} \right)^+ \mathbf{1}_{\{\Lambda_T \leq b\}} + (\Lambda_T - 1) \, \mathbf{1}_{\{\Lambda_T > b\}} \right\} \right] \\
&= \mathbb{E}_{\mathbb{Q}} \left[\mathrm{e}^{-rT} L_T \left\{ w \left(\Lambda_T - \frac{1}{w} \right)^+ \right. \right. \\
&\quad \left. \left. - (w\, \Lambda_T - 1) \, \mathbf{1}_{\{\Lambda_T > b\}} + (\Lambda_T - 1) \, \mathbf{1}_{\{\Lambda_T > b\}} \right\} \right] \\
&= \mathbb{E}_{\mathbb{Q}} \left[\mathrm{e}^{-rT} L_T \left\{ w \left(\Lambda_T - \frac{1}{w} \right)^+ + (1 - w)\, \Lambda_T \, \mathbf{1}_{\{\Lambda_T > b\}} \right\} \right].
\end{aligned} \tag{3}$$

The main objective function within this model is to maximize V_0^{SR} with respect to the insurer's risk $\mathbf{\Sigma} = (\sigma_L,\, \sigma_\Lambda)$. The insurer can adjust the value of risk $\mathbf{\Sigma}$ to the optimal value $\mathbf{\Sigma}^* = \arg\max_{\mathbf{\Sigma}} V_0^{SR}$ by:

- using reinsurance to control underwriting risk σ_L, or
- using both reinsurance and optimal investment strategies to control the volatility σ_Λ of the solvency ratio.

2.2 The RBC model

We consider the insurer's underwriting process over the period of time $[0, T]$ between two consecutive regulatory's audits and define insurer's solvency within the RBC model in terms of risk based capital and dynamics of the company's surplus S as a stochastic process $S_t,\ t \geq 0$. Let $S^* \leq S_0$ denote the minimal capitalization level (regulatory surplus or risk based capital) at which the insurer is considered financially solvent by an insurance regulator, and the insurer will be announced insolvent at time $\tau = \inf\{t \in \{T, 2T, ...\} : S_t < S^*\}$. It is assumed that at time τ the insurer's surplus will be fairly divided among its creditors, so its surplus will immediately drop to zero (the company is liquidated), and when the insurer is solvent on the maturity date, it retains the excess of minimal level of surplus (regulatory capital) at the constant retention rate and the rest is paid to shareholders as dividends. So the maximization of the terminal value of dividends is equivalent to the maximization of the expected terminal value of the company's surplus net of regulatory surplus. In contrast to the SR model, where we assumed that all the insurer's liabilities are paid at the end of the period, the liabilities within the RBC model are assumed to be paid continuously over the period. We define the insurer's economic state as *financially distressed* in the following way. If the insurer's value of its surplus S never hits a pre-specified financial distress barrier, say D, over the period $[0, T]$, the terminal surplus value is S_T. If the financial distress barrier is hit, the insurer incurs deadweight losses and the terminal surplus value falls to $F(S_T) < S_T$. Here, the function F represents the terminal surplus value net of FD costs. Let m_T^S denote the minimum value of surplus over $[0, T]$ (i.e. $m_T^S = \min\limits_{t \in [0,T]} S_t$). We assume that the insurer experiences financial distress when its surplus hits a pre-specified level which is lower than regulatory surplus, i.e. $D < S^*$. If the insurance company was not financially distressed ($m_T^S > D$) and it is solvent on the terminal date ($S_T > S^*$), then the terminal value of the excess of the minimal level of the company's surplus is $S_T - S^*$. The terminal value of the excess of minimal level of surplus is equal to $F(S_T) - S^* > 0$ in the case when the company was financially distressed but it remains solvent on the maturity date. From here we can calculate the expected present value of the company's surplus net of regulatory capital and FD costs:

$$\begin{aligned}
V_0^{RBC} &= \mathrm{e}^{-rT} \mathbb{E}_{\mathbb{Q}} \left[(S_T - S^*)^+ \mathbf{1}_{\{\text{FinHealthy}\}} \right. \\
&\quad \left. + (F(S_T) - S^*)^+ \mathbf{1}_{\{\text{FinDistressed \& Solvent}\}} \right] \\
&= \mathrm{e}^{-rT} \mathbb{E}_{\mathbb{Q}} \left[(S_T - S^*)^+ \, \mathbf{1}_{\{m_T^S > D\}} + (F(S_T) - S^*)^+ \, \mathbf{1}_{\{m_T^S \leq D\}} \right],
\end{aligned} \tag{4}$$

where $\mathbb{Q}$ is an equivalent martingale measure, that exists under the assumption that the market for assets and liabilities is arbitrage free, but incomplete (as it is typical in insurance). The value V_0^{RBC} in (4) can be rewritten in the following way

$$\begin{aligned}
V_0^{RBC} &= \mathrm{e}^{-rT} \mathbb{E}_{\mathbb{Q}} \left[(S_T - S^*)^+ \, \mathbf{1}_{\{m_T^S > D\}} + (F(S_T) - S^*)^+ \, \mathbf{1}_{\{m_T^S \leq D\}} \right] \\
&= \mathrm{e}^{-rT} \mathbb{E}_{\mathbb{Q}} \left[(S_T - S^*)^+ - \left[(S_T - S^*)^+ - (F(S_T) - S^*)^+ \right] \, \mathbf{1}_{\{m_T^S \leq D\}} \right] \\
&= \mathrm{e}^{-rT} \mathbb{E}_{\mathbb{Q}} \Big[(S_T - S^*) - (S_T - F(S_T)) \, \mathbf{1}_{\{m_T^S \leq D\} \cap \{F(S_T) > S^*\}} \\
&\quad + (S^* - S_T) \left(\mathbf{1}_{\{S_T \leq S^*\}} + \mathbf{1}_{\{m_T^S \leq D\} \cap \{S_T > S^* \geq F(S_T)\}} \right) \Big]
\end{aligned} \tag{5}$$

We can see in (5) that the value V_0 has three components. The first term $\mathbb{E}_{\mathbb{Q}}[(S_T - S^*)]$ represents the unconstrained surplus value of the company net of regulatory capital. The second term $\mathbb{E}_{\mathbb{Q}}\left[(S_T - F(S_T))\, \mathbf{1}_{\{m_T^S \leq D\} \cap \{F(S_T) > S^*\}}\right]$ represents the deadweight losses caused by financial distress. The shareholders of a financially distressed but solvent insurance company bear financial distress costs and thus the terminal value of surplus net of regulatory capital is reduced by this amount. These costs can be reduced by risk management strategies. Existence of the third positive term in (5) can be increased by risk-taking strategies by shareholders. By increasing the company's integrated investment-underwriting risk, the shareholders can make themselves better off by increasing the call option value. But at the same time the expected losses in the event of financial distress also increases with the risk. Therefore, the optimal level of integrated investment-underwriting risk is determined by the trade-off between these two incentives. An insurer can use two possibilities to control investment risk:
1) it can construct an investment portfolio with optimal volatility; or
2) the volatility can be fixed and an insurer can reduce risk by buying derivative contracts such as options. The underwriting risk can be reduced by traditional *cession* (risk transfer), i.e. through purchase of reinsurance contracts in the reinsurance market. It is worth noticing that to some extent the reinsurance contract resembles derivative contracts for financial risk. Using both, an agent can reduce risk (volatility of investment portfolio or underwriting risk, which is traditionally measured by the probability of insolvency). So far we have considered the insurer's surplus in the general form of a stochastic process. To be able to calculate the value V_0^{RBC} in (5) we have to be more specific about the model of the company's surplus. The following defines the dynamics of the company's surplus under the physical probability measure $\mathbb{P}$.

Dynamic model of the insurance company's surplus. We consider the diffusion model of an insurer's surplus formulated by Powers in (1995 [16]). In practice insurance firms receive insurance premiums and must establish liability accounts for the unearned premium reserves (unexpired risk reserve) and the loss (outstanding claims) reserve (see Outreville (1998 [15])). The unexpired risk reserve can be used to fund a surrender value to the policyholder in the case of withdrawal from the insurance policy. The loss reserve is the reserve to pay anticipated losses, including incurred but not reported (IBNR) losses. In practice these losses create the company's main liabilities. An insurance company's assets consist of its invested funds arising from premiums and current value of surplus. These assets are invested to generate income. The underwriting profit can be estimated by deducting the current value of loss reserves and earned expenses from earned premiums. The insurance company's expenses may consists of acquisition costs that are approximately proportional to premiums and administrative costs that are approximately proportional to current loss reserves. On the one hand the insurance liabilities (reserves) are increased by instantaneous policy writing and at the same time are decreased by the instantaneous loss payment outflow. On the other hand, assets are increased by instantaneous earned premiums and the investment income inflow but are decreased by instantaneous loss payments. We will use the following notation:

- S_t denotes the surplus of the insurance company at time t;
- L_t denotes the expected current loss reserves at time t;
- dP_t denotes the instantaneous earned premium inflow at time t.

Let π_L be the underwriting profit and expense loading charged by the insurer, expressed as a proportion of expected losses, and

$$dP_t = (1+\pi_L)L_t dt,$$

that is $1+\pi_L$ is the gross insurance premium rate. Then the instantaneous underwriting profit at time t follows the diffusion process

$$\begin{aligned} d\Pi_t &= dP_t - L_t dt - \varepsilon_L L_t dt - \varepsilon_P dP_t + s_L dW_t^L \\ &= (\pi_L(1-\varepsilon_P) - (\varepsilon_L + \varepsilon_P))\, L_t dt + s_L dW_t^L, \end{aligned}$$

where $\varepsilon_L > 0$ and $\varepsilon_P > 0$ denote, respectively, the expense ratio for administrative expenses that are proportional to expected losses and the expense ratio for acquisition expenses that are proportional to premiums; the volatility of this diffusion is assumed to be $s_L = \sigma_L L_t$. By definition the insurer's assets are equal to the insurance liabilities (loss reserves) plus surplus, and thus investment income is generated by investing these assets in the capital market. It is assumed that the total investment return follows a geometric Brownian motion with drift r_I and volatility σ_I, i.e. the instantaneous investment income inflow at time t is

$$dI_t = r_I(L_t + S_t)dt + \sigma_I(L_t + S_t)dW_t^I.$$

Denote by $\lambda_t = \frac{L_t}{S_t}$ the current leverage ratio at time t. For the sake of simplicity and ability to illustrate results we assume that W^I and W^L are independent, the insurer can control its leverage ratio over the period to keep it constant at level λ. Now, taking into account the fact that the instantaneous insurer's surplus at time t, dS_t, is the sum of instantaneous values of investment income and underwriting profit (i.e. $d(I_t + \Pi_t)$) we can write the diffusion process of the insurer's surplus as

$$\begin{aligned} dS_t &= d(I_t + \Pi_t) = (\pi_L(1-\varepsilon_P) - (\varepsilon_L + \varepsilon_P))\, L_t dt \\ &\quad + r_I(1+\lambda)S_t dt + \sigma_L L_t dW_t^L + \sigma_I(1+\lambda)S_t dW_t^I \\ &= \mu S_t dt + \sigma\, S_t dW_t, \end{aligned} \tag{6}$$

where $\mu = \lambda\left(\pi_L(1-\varepsilon_P) - (\varepsilon_L + \varepsilon_P) + r_I\right) + r_I$, $\sigma = \sqrt{\lambda^2\sigma_L^2 + (\lambda+1)^2\sigma_I^2}$, and W is a standard Brownian motion associated with the surplus. So, we come up with a model of the insurer's surplus described by a geometric Brownian motion under the physical probability measure $\mathbb{P}$. This means that within the RBC model the company's surplus is always positive. This is a difference between this and the SR model, where the surplus, defined as a difference between the values of assets and liabilities, can take negative values in the case of insolvency. However, it should be noticed that the use of the log-normal model of the surplus can be meaningful within the RBC model, since the insurer's insolvency occurs at a positive level of capitalization, or in other words, it is defined by the event where the insurer's surplus hits a positive threshold, i.e. value of the regulatory surplus. Similarly to the SR model we define the objective function to be $\max_\sigma V_T^{RBC}$.

Again, the insurer can adjust the value of risk σ to the optimal value $\sigma^* = \arg\max_\sigma V_0^{RBC}$ by:

- using reinsurance to control the underwriting risk σ_L, and/or
- using optimal investment strategies to control the investment risk σ_I.

In the case of taking quota share proportional reinsurance with the cedent's retention level $\alpha \in (0,1)$ the drift and diffusion of S in (6) are respectively equal

$\mu = \alpha\lambda\left[\pi_L(1-\varepsilon_P) - (\varepsilon_L + \varepsilon_P) + r_I - p_{re}(1-\alpha)\right] + r_I$ and
$\sigma = \sqrt{\alpha^2\lambda^2\sigma_L^2 + (\alpha\lambda+1)^2\sigma_I^2}$, where p_{re} is the reinsurance premium rate.

3 Risk management incentives in the SR and RBC models of the maximization of shareholders' value in the presence of FD costs

In this subsection we calculate the value of the objective function (insurer value) in the SR and RBC models of the maximization of shareholders' value in the presence of FD costs. To illustrate the existence of risk management incentives in insurance in the presence of FD costs, we provide an analytical form of insurer value in the RBC model with FD costs that come in the form of lost upside potential of the surplus. We show that the insurer value is a function of insurance company's risk, FD barrier, initial level of capitalization, required minimal level of capitalization and parameter of FD costs. We determine the optimal value of insurance company's risk that maximizes the insurer value, and investigate the company's risk management sensitivities with respect to parameter of FD costs, FD barrier and initial level of capitalization.

3.1 The main results of the SR model

In order to calculate the value V_0^{SR} in (3) we change the numéraire from the risk free bank account to L_t and correspondingly change the measure from the risk-neutral measure $\mathbb{Q}$, defined under the old numéraire, to the new one $\widetilde{\mathbb{Q}}$. Under the risk-neutral measure $\mathbb{Q}$

$$L_T = L_0 \mathrm{e}^{\mu_L T}\, Z_T,$$

where $Z_T = \mathrm{e}^{\sigma_L W_T^L - \frac{1}{2}\sigma_L^2 T}$ is a martingale with respect to $\mathbb{Q}$. The new risk-neutral measure $\widetilde{\mathbb{Q}}$ is defined as follows

$$\left.\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}}\right| \mathcal{F}_T = Z_T \;\text{ and }\; \forall\, t \in [0,T]\; \mathbb{E}_{\mathbb{Q}}\left[Z_T \,|\, \mathcal{F}_t\right] = Z_t.$$

Then for any function of the terminal value of Λ_T using Girsanov's theorem we obtain

$$\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[\mathrm{e}^{-r(T-t)}\, f(\Lambda_T)\middle|\, \mathcal{F}_t\right] = \mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-r(T-t)}\frac{Z_T}{Z_t} f(\Lambda_T)\middle|\, \mathcal{F}_t\right]$$

and thus

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-r(T-t)} L_T\, f(\Lambda_T)\middle|\, \mathcal{F}_t\right] &= \mathbb{E}_{\widetilde{\mathbb{Q}}}\left[\mathrm{e}^{-r(T-t)}\, f(\Lambda_T)\middle|\, \mathcal{F}_t\right] Z_t\, L_0\, \mathrm{e}^{\mu_L T} \\ &= L_t\, \mathrm{e}^{\mu_L (T-t)} \mathbb{E}_{\widetilde{\mathbb{Q}}}\left[\mathrm{e}^{-r(T-t)}\, f(\Lambda_T)\middle|\, \mathcal{F}_t\right].\end{aligned}$$

Now, using Levy's Theorem on martingale characterization of Brownian motion (see Karatzas and Shreve (1988 [10]) and also see Lemma 1 in Appendix A in Sherris and van der Hoek (2006 [17])) it can be shown that

$$\widetilde{W}_t^{\Lambda} = W_t^{\Lambda} - \rho_{L\Lambda}\, \sigma_L$$

is a $\widetilde{\mathbb{Q}}$-Brownian motion, where $\rho_{L\Lambda}$ is the instantaneous correlation between the value of liabilities and solvency ratio, defined by

$$dW_t^L\, dW_t^{\Lambda} = \rho_{L\Lambda}\, dt.$$

Therefore, the new $\widetilde{\mathbb{Q}}$-risk-neutral dynamics of Λ are

$$\begin{aligned} d\Lambda_t &= \mu_\Lambda\, \Lambda_t\, dt + \sigma_\Lambda\, \Lambda_t\, dW_t^{\Lambda} \\ &= \mu_\Lambda\, \Lambda_t\, dt + \sigma_\Lambda\, \Lambda_t\, \left(d\widetilde{W}_t^{\Lambda} + \rho_{L\Lambda}\, dt\right) \\ &= \widetilde{\mu}_\Lambda\, \Lambda_t\, dt + \widetilde{\sigma}_\Lambda\, \Lambda_t\, d\widetilde{W}_t^{\Lambda} \\ &= \left(r - (r - \widetilde{\mu}_\Lambda)\right) \Lambda_t\, dt + \widetilde{\sigma}_\Lambda\, \Lambda_t\, d\widetilde{W}_t^{\Lambda}, \end{aligned} \tag{7}$$

where $\widetilde{\mu}_\Lambda = \mu_\Lambda + \rho_{L\Lambda}\, \sigma_\Lambda\, \sigma_L$ and $\widetilde{\sigma}_\Lambda = \sigma_\Lambda$. Now, the expected present value V_0^{SR} of shareholders' terminal payoff in (3) can be rewritten in the following way

$$\begin{aligned} V_0^{SR} &= \mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-rT}\, L_T\, \left\{ w\, \left(\Lambda_T - \frac{1}{w}\right)^+ + (1-w)\, \Lambda_T\, \mathbf{1}_{\{\Lambda_T > b\}} \right\}\right] \\ &= L_0\, \mathrm{e}^{\mu_L T} \left\{ w\, \mathbb{E}_{\widetilde{\mathbb{Q}}}\left[\mathrm{e}^{-rT}\left(\Lambda_T - \frac{1}{w}\right)^+\right]\right. \\ &\qquad \left. + (1-w)\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[\mathrm{e}^{-rT}\Lambda_T\, \mathbf{1}_{\{\Lambda_T > b\}}\right] \right\}. \end{aligned} \tag{8}$$

According to the $\widetilde{\mathbb{Q}}$-risk-neutral dynamics of Λ in (7) the value

$$c_0 = \mathbb{E}_{\widetilde{\mathbb{Q}}}\left[\mathrm{e}^{-rT}\, (\Lambda_T - K)^+\right]$$

is the value at time 0 of a European call option on an underlying asset paying a continuous dividend at rate $r - \widetilde{\mu}_\Lambda$ with current price Λ_0, exercise time T, and a strike price K. Using the classical Black-Scholes result we obtain

$$c_0 = \Lambda_0\, \mathrm{e}^{-(r-\widetilde{\mu}_\Lambda)T}\, \Phi\left(d_+\right) - K\, \mathrm{e}^{-rT}\, \Phi\left(d_-\right), \tag{9}$$

where $d_\pm = \dfrac{\ln \frac{\Lambda_0}{K} + \left(\widetilde{\mu}_\Lambda \pm \frac{\sigma_\Lambda^2}{2}\right) T}{\sigma_\Lambda \sqrt{T}}$ and Φ is the standard normal cdf. Using (9) we can immediately calculate the expectation in the first term in (8) and also the expectation in the second term, which is equal to the first term in (9) with $K = b$. So,

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\begin{aligned}
V_0^{SR} &= L_0\, \mathrm{e}^{\mu_L T} \left\{ w\, \mathbb{E}_{\widetilde{\mathbb{Q}}} \left[\mathrm{e}^{-rT} \left(\Lambda_T - \frac{1}{w} \right)^+ \right] + (1-w) \mathbb{E}_{\widetilde{\mathbb{Q}}} \left[\mathrm{e}^{-rT} \Lambda_T\, \mathbf{1}_{\{\Lambda_T > b\}} \right] \right\} \\
&= L_0\, \mathrm{e}^{\mu_L T} \Big\{ w\, \Lambda_0\, \mathrm{e}^{-\left(r - \widetilde{\mu}_\Lambda\right) T}\, \Phi\left(h_+\right) - \mathrm{e}^{-rT}\, \Phi\left(h_-\right) \\
&\quad + (1-w) \Lambda_0\, \mathrm{e}^{-\left(r - \widetilde{\mu}_\Lambda\right) T}\, \Phi\left(l_+\right) \Big\} \\
&= L_0 \left\{ w\, \Lambda_0\, \mathrm{e}^{\widetilde{\mu}_\Lambda T}\, \Phi\left(h_+\right) - \Phi\left(h_-\right) + (1-w) \Lambda_0\, \mathrm{e}^{\widetilde{\mu}_\Lambda T}\, \Phi\left(l_+\right) \right\},
\end{aligned} \tag{10}
$$

where $h_\pm = \frac{\ln(w\, \Lambda_0) + \left(\widetilde{\mu}_\Lambda \pm \frac{\sigma_\Lambda^2}{2}\right) T}{\sigma_\Lambda \sqrt{T}}$ and $l_+ = \frac{\ln \frac{\Lambda_0}{b} + \left(\widetilde{\mu}_\Lambda + \frac{\sigma_\Lambda^2}{2}\right) T}{\sigma_\Lambda \sqrt{T}}$
(we also used the fact that $\mu_L = r$). From (10) we can see that the value V_0^{SR} depends on FD costs, the FD barrier and the parameters μ_L, σ_L and σ_Λ. Taking the partial derivative of V_0^{SR} with respect to w for $\frac{1}{b} < w < 1$ we obtain

$$
\frac{\partial}{\partial w} V_0^{SR} = L_0\, \Lambda_0\, \mathrm{e}^{\widetilde{\mu}_\Lambda T} \left(\Phi\left(h_+\right) - \Phi\left(l_+\right) \right) > 0, \text{ since } h_+ > l_+.
$$

This means that the value V_0^{SR} increases with a decrease in the FD costs (or with an increase in w). It is easy to see that the third term in (10) decreases when the FD barrier b increases. Therefore, the value V_0^{SR} decreases with an increase in the FD barrier b. We also consider the problem of finding an optimal volatility σ_Λ^* of the solvency ratio under which the value V_0^{SR} takes its maximum. To do so we investigate the FOC

$$
\begin{aligned}
\frac{\partial}{\partial \sigma_\Lambda} V_0^{SR} &= \rho_{L\,\Lambda}\, \sigma_L\, L_0\, T \Lambda_0\, \mathrm{e}^{\widetilde{\mu}_\Lambda T} \left[w \Phi\left(h_+\right) + (1-w) \Phi\left(l_+\right) \right] \\
&\quad + L_0 \left(w\, \Lambda_0\, \mathrm{e}^{\widetilde{\mu}_\Lambda T}\, \phi\left(h_+\right) \frac{\partial h_+}{\partial \sigma_\Lambda} - \phi\left(h_-\right) \frac{\partial}{\partial \sigma_\Lambda} \left(h_+ - \sigma_\Lambda \sqrt{T} \right) \right) \\
&\quad + L_0\, (1-w) \Lambda_0\, \mathrm{e}^{\widetilde{\mu}_\Lambda T}\, \phi\left(l_+\right) \frac{\partial l_+}{\partial \sigma_\Lambda} \\
&= \rho_{L\,\Lambda}\, \sigma_L\, L_0\, T \Lambda_0\, \mathrm{e}^{\widetilde{\mu}_\Lambda T} \left[w \Phi\left(h_+\right) + (1-w) \Phi\left(l_+\right) \right] + L_0 \sqrt{T}\, \phi\left(h_-\right) \\
&\quad + L_0\, (1-w) \left(\frac{\sqrt{T}}{2} - \frac{\ln \frac{\Lambda_0}{b} + \mu_\Lambda T}{\sigma_\Lambda^2 \sqrt{T}} \right) \Lambda_0\, \mathrm{e}^{\widetilde{\mu}_\Lambda T}\, \phi\left(l_+\right),
\end{aligned}
$$

where we used the fact that $\frac{\phi(h_-)}{\phi(h_+)} = w\, \Lambda_0\, \mathrm{e}^{\widetilde{\mu}_\Lambda T}$. Taking into account the fact that a higher value of liabilities corresponds to a lower value of the solvency ratio, we conclude that the instantaneous correlation $\rho_{L\,\Lambda}$ between the liabilities and solvency ratio is negative. Under this condition, the FOC equation

$$
\frac{\partial}{\partial \sigma_\Lambda} V_0^{SR} = 0
$$

may have a solution for optimal σ_Λ. The FOC equation is non-linear with respect to the σ_Λ and can not be solved analytically. In order to obtain results, numerical methods must be used to find an optimal value σ_Λ^*.

3.2 The main results of the RBC model

We consider the following three forms of financial distress costs:

1) deadweight losses are proportional to the terminal value of the insurer's surplus with proportionate coefficient $1 - w,\ w \in (0, 1)$, and
2) deadweight losses are a constant amount C
3) deadweight losses are of the form of lost upside potential of terminal value of company's surplus.

For the first form of FD costs the value of the function F at S_T as the terminal value of the company's excess of regulatory surplus net of FD costs is equal to

$$F_1(S_T) \triangleq w\, S_T,$$

for the second form it is

$$F_2(S_T) \triangleq S_T - C,$$

and for the third form it is

$$F_3(S_T) \triangleq S_T - (S_T - (S^* + U))^+,$$

where $U > 0$ is the parameter of FD costs. The higher the value of U, the higher value of upside potential of S_T, or, equivalently, the lower value of FD costs. We rewrite the value V_0^{RBC} from (5) for all the three forms of FD costs. In the first case where $F = F_1$

$$V_0^{RBC} \triangleq V_0^{(1)} = \mathrm{e}^{-rT}\mathbb{E}_{\mathbb{Q}}\left[(S_T - S^*)^+\,\mathbf{1}_{\{m_T^S > D\}} + w\left(S_T - \frac{S^*}{w}\right)^+\mathbf{1}_{\{m_T^S \le D\}}\right], \tag{11}$$

in the second case where $F = F_2$

$$\begin{aligned} V_0^{RBC} &\triangleq V_0^{(2)} \\ &= \mathrm{e}^{-rT}\,\mathbb{E}_{\mathbb{Q}}\left[(S_T - S^*)^+\,\mathbf{1}_{\{m_T^S > D\}} + (S_T - (C + S^*))^+\,\mathbf{1}_{\{m_T^S \le D\}}\right] \\ &= \mathrm{e}^{-rT}\,\mathbb{E}_{\mathbb{Q}}\Big[(S_T - S^*)^+ - (S_T - S^*)^+\,\mathbf{1}_{\{m_T^S \le D\}} \\ &\quad + (S_T - (C + S^*))^+\,\mathbf{1}_{\{m_T^S \le D\}}\Big], \end{aligned} \tag{12}$$

and in the third case where $F = F_3$

$$\begin{aligned} V_0^{RBC} \triangleq V_0^{(3)} = \mathrm{e}^{-rT}\,\mathbb{E}_{\mathbb{Q}}\Big[&(S_T - S^*)^+\,\mathbf{1}_{\{m_T^S > D\}} \\ &+ (S_T - S^*)^+\,\mathbf{1}_{\{m_T^S \le D\} \cap \{S_T \le S^* + U\}} + U\,\mathbf{1}_{\{m_T^S \le D\} \cap \{S_T > S^* + U\}}\Big]. \end{aligned} \tag{13}$$

As we can see in all three cases the value V_0^{RBC} is represented by the values of two different barrier options. In the third case there is a term representing the joint distribution of the minima and the terminal value of the company's surplus that follows a geometric Brownian motion. To evaluate V_0^{RBC} we will use the well known approach for the pricing of path-dependent options or barrier options (e.g. see Etheridge (2002 [6]) or Musiela and Rutkowski (1998 [13])). We know that under the equivalent martingale measure $\mathbb{Q}$, $S_t = S_0 \mathrm{e}^{\sigma Y_t}$, where $Y_t = \frac{r-\frac{1}{2}\sigma^2}{\sigma}\, t + W_t^{\mathbb{Q}}$ and $W_t^{\mathbb{Q}}$ is a $\mathbb{Q}$-Brownian motion. It is also well known from financial calculus that for $Y_t = bt + W_t^{\mathbb{Q}}$, and $m_t = \min\limits_{s\in[0,t]} Y_s$

$$\mathbb{Q}\left[m_T \le a,\, Y_T \in dx\right] = \begin{cases} p_T(bT,\, x)dx, & x < a, \\ \mathrm{e}^{2ab} p_T(2a + bT,\, x)dx, & x \ge a, \end{cases}$$

where $p_t(x,y) = \frac{1}{\sqrt{2\pi t}}\,\mathrm{e}^{-\frac{(x-y)^2}{2t}}$ is the Brownian transition density function (see Etheridge (2002 [6]) pp. 144-148). The value of V_0^{RBC} in (11) is

$$\begin{aligned} V_0^{(1)} = {} & \mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-rT}\left(S_T - S^*\right)^+\right] - \mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-rT}\left(S_T - S^*\right)^+ \mathbf{1}_{\{m_T^S \le D\}}\right] \\ & + w\, \mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-rT}\left(S_T - \frac{S^*}{w}\right)^+ \mathbf{1}_{\{m_T^S \le D\}}\right], \end{aligned}$$

and in (12) it is

$$\begin{aligned} V_0^{(2)} = {} & \mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-rT}\left(S_T - S^*\right)^+\right] - \mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-rT}\left(S_T - S^*\right)^+ \mathbf{1}_{\{m_T^S \le D\}}\right] \\ & + \mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-rT}\left(S_T - (C + S^*)\right)^+ \mathbf{1}_{\{m_T^S \le D\}}\right] \end{aligned}$$

i.e. in the first two cases it is equal to the value of a call option plus the difference of two *down-and-in-call* options with different strike prices. For any strike price $K > 0$ and $a \triangleq \frac{1}{\sigma}\ln\left(\frac{D}{S_0}\right)$ the value of a *down-and-in-call* option is

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-rT}\left(S_T - K\right)^+ \mathbf{1}_{\{m_T^S \le D\}}\right] = \mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-rT}\left(S_T - K\right)^+ \mathbf{1}_{\left\{m_T \le \frac{1}{\sigma}\ln\left(\frac{D}{S_0}\right)\right\}}\right] \\ & = \mathrm{e}^{-rT} \int\limits_{\frac{1}{\sigma}\ln\left(\frac{K}{S_0}\right)}^{\infty} \left(S_0 \mathrm{e}^{\sigma x} - K\right) \mathbb{Q}\left[m_T \le a,\, Y_T \in dx\right] \\ & = \mathrm{e}^{-rT} \int\limits_{\frac{1}{\sigma}\ln\left(\frac{K}{S_0}\right)}^{\infty} \left(S_0 \mathrm{e}^{\sigma x} - K\right) \mathrm{e}^{2ab} p_T(2a + bT, x)\, dx = \left(\frac{D}{S_0}\right)^{\frac{2r}{\sigma^2}-1} \mathbf{C}\left(\frac{D^2}{S_0}, 0, K, T\right) \\ & = \left(\frac{D}{S_0}\right)^{\frac{2r}{\sigma^2}-1} \left[\frac{D^2}{S_0}\,\Phi\left(\frac{\ln\left(\frac{D^2}{KS_0}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - K\,\mathrm{e}^{-rT}\,\Phi\left(\frac{\ln\left(\frac{D^2}{KS_0}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)\right], \end{aligned}$$

where $\mathbf{C}\left(x, t, K, T\right)$ is the price at time t of a European call option with strike price K, maturity time T and the stock price x at time t. Therefore,

$$
\begin{aligned}
V_0^{(1)} &= \mathbf{C}\left(S_0, 0, S^*, T\right) - \left(\frac{D}{S_0}\right)^{\frac{2r}{\sigma^2}-1} \left[\mathbf{C}\left(\frac{D^2}{S_0}, 0, S^*, T\right) - w\,\mathbf{C}\left(\frac{D^2}{S_0}, 0, \frac{S^*}{w}, T\right)\right], \\
V_0^{(2)} &= \mathbf{C}\left(S_0, 0, S^*, T\right) - \left(\frac{D}{S_0}\right)^{\frac{2r}{\sigma^2}-1} \left[\mathbf{C}\left(\frac{D^2}{S_0}, 0, S^*, T\right) - \mathbf{C}\left(\frac{D^2}{S_0}, 0, C + S^*, T\right)\right].
\end{aligned}
$$

Using the same approach of barrier options valuation we can calculate the value $V_0^{(3)}$. To find an optimal $\widehat{\sigma}$ under which the value function V achieves its maximum one has to solve FOC. In general case where $r \neq 0$ it is difficult to obtain explicit form of optimal σ and the numerical methods must be used. However, using change of the numéraire we can derive an analytical form of optimal σ for some cases. To illustrate and support our initial hypothesis of existence of insurance risk management incentives under the present of FD costs we will evaluate the shareholders value for the third form of FD costs and $r = 0$. According to the Numéraire Invariance Theorem (see Geman et al (1995 [8]) or Duffie (2001 [5])) there exists a sequence of pairs: a numéraire $N_t^{(i)} = \mathrm{e}^{r_i\, t}$ (with $r_{i+1} < r_i$ for all i) and a probability measure $\mathbb{Q}_{N^{(i)}}$, equivalent to the initial measure $\mathbb{Q}$, such that $\mathbb{E}_{\mathbb{Q}_{N^{(i)}}}\left[\frac{f(S_T)}{N_T^{(i)}}\right] = \mathbb{E}_{\mathbb{Q}}\left[\frac{f(S_T)}{N_0^{(i)}\,\mathrm{e}^{rT}}\right]$. Taking $r_i \to 0$ we obtain a risk-neutral measure $\mathbb{Q}_{N^{(\infty)}} \triangleq \mathbb{Q}_0$ such that $\mathbb{E}_{\mathbb{Q}_0}\left[f(S_T)\right] = \mathbb{E}_{\mathbb{Q}}\left[\frac{f(S_T)}{\mathrm{e}^{rT}}\right]$.

$$
\begin{aligned}
V_0^{(3)} = {} & \mathbb{E}_{\mathbb{Q}_0}\Big[(S_T - S^*)^+\,\mathbf{1}_{\{m_T^S > D\}} + (S_T - S^*)^+\,\mathbf{1}_{\{m_T^S \le D\} \cap \{S_T \le S^* + U\}} \\
& + U\,\mathbf{1}_{\{m_T^S \le D\} \cap \{S_T > S^* + U\}}\Big] = \mathbb{E}_{\mathbb{Q}_0}\Big[(S_T - S^*)^+ - (S_T - S^*)^+\,\mathbf{1}_{\{m_T^S \le D\}}\Big] \\
& + \mathbb{E}_{\mathbb{Q}_0}\Big[(S_T - S^*)^+\,\mathbf{1}_{\{m_T^S \le D\}} - (S_T - S^*)^+\,\mathbf{1}_{\{m_T^S \le D\} \cap \{S_T > S^* + U\}}\Big] \\
& + \mathbb{E}_{\mathbb{Q}_0}\Big[U\,\mathbf{1}_{\{m_T^S \le D\} \cap \{S_T > S^* + U\}}\Big] = \mathbb{E}_{\mathbb{Q}_0}\left[(S_T - S^*)^+\right] \\
& - \mathbb{E}_{\mathbb{Q}_0}\Big[(S_T - S^*)\,\mathbf{1}_{\{m_T^S \le D\} \cap \{S_T > S^* + U\}}\Big] \\
& + U\,\mathbb{Q}_0\left[\{m_T^S \le D\} \cap \{S_T > S^* + U\}\right] \\
= {} & \mathbb{E}_{\mathbb{Q}_0}\left[(S_T - S^*)^+\right] - \mathbb{E}_{\mathbb{Q}_0}\Big[(S_T - (S^* + U))\,\mathbf{1}_{\{m_T^S \le D\} \cap \{S_T > S^* + U\}}\Big] \\
= {} & \mathbf{C}\left(S_0, 0, S^*, T\right) - \mathbf{C}\left(D, 0, \frac{S_0(S^* + U)}{D}, T\right).
\end{aligned}
$$

To find an extremum $\widehat{\sigma}$ of the function $V_0(\sigma)$ first we have to calculate the derivative of $\mathbf{C}\left(x, t, K, T\right) = x\,\Phi\left(d_+\right) - K\,\Phi\left(d_-\right)$.

$$
\begin{aligned}
\frac{\partial}{\partial\sigma}\mathbf{C}\left(x, 0, K, T\right) &= \frac{\partial}{\partial\sigma}\left\{x\,\Phi\left(\frac{\ln\frac{x}{K} + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right) - K\,\Phi\left(\frac{\ln\frac{x}{K} - \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right)\right\} \\
&= x\,\phi(d_+)\frac{\partial d_+}{\partial\sigma} - K\,\phi(d_-)\frac{\partial d_-}{\partial\sigma} \\
&= x\,\phi(d_+)\frac{\partial d_+}{\partial\sigma} - K\,\phi(d_-)\left(\frac{\partial d_+}{\partial\sigma} - \sqrt{T}\right) \\
&= x\,\phi(d_+)\frac{\partial d_+}{\partial\sigma} - x\,\phi(d_+)\left(\frac{\partial d_+}{\partial\sigma} - \sqrt{T}\right) \\
&= x\,\phi(d_+)\sqrt{T} = x\,\phi\left(\frac{\ln\frac{x}{K} + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right)\sqrt{T} \qquad (14)
\end{aligned}
$$

where we have used the relationship $\frac{\phi(d_-)}{\phi(d_+)} = \frac{x}{K}$. Using (14) we obtain

$$\frac{\partial}{\partial\sigma}V_0^{(1)} = S_0\phi\left(\frac{\ln\frac{S_0}{S^*}+\frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right)\sqrt{T} - \frac{S_0}{D}\left[\frac{D^2}{S_0}\,\phi\left(\frac{\ln\frac{D^2}{S_0S^*}+\frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right)\sqrt{T}\right.$$
$$\left. -w\frac{D^2}{S_0}\,\phi\left(\frac{\ln\frac{wD^2}{S_0S^*}+\frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right)\sqrt{T}\right];$$

$$\frac{\partial}{\partial\sigma}V_0^{(2)} = S_0\phi\left(\frac{\ln\frac{S_0}{S^*}+\frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right)\sqrt{T} - \frac{S_0}{D}\left[\frac{D^2}{S_0}\,\phi\left(\frac{\ln\frac{D^2}{S_0S^*}+\frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right)\sqrt{T}\right.$$
$$\left. -\frac{D^2}{S_0}\,\phi\left(\frac{\ln\frac{D^2}{S_0(C+S^*)}+\frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right)\sqrt{T}\right];$$

$$\frac{\partial}{\partial\sigma}V_0^{(3)} = S_0\phi\left(\frac{\ln\frac{S_0}{S^*}+\frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right)\sqrt{T} - D\,\phi\left(\frac{\ln\frac{D^2}{S_0(S^*+U)}+\frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right)\sqrt{T}.$$

The corresponding score equation $\frac{\partial}{\partial\sigma}V_0^{(3)} = 0$ can be rearranged to the linear one with respect to σ^2:

$$\frac{\partial}{\partial\sigma}V_0^{(3)} = 0 \;\Leftrightarrow\; \left(\ln\frac{S_0}{S^*}+\frac{\sigma^2}{2}T\right)^2 - 2\sigma^2T\ln S_0 = \left(\ln\frac{D^2}{S_0(S^*+U)}+\frac{\sigma^2}{2}T\right)^2 - 2\sigma^2T\ln D$$
$$\Leftrightarrow\; \widehat{\sigma}^2 = \frac{1}{T}\frac{\ln\frac{D^2S^*}{S_0^2(S^*+U)}\ln\frac{D^2}{S^*(S^*+U)}}{\ln\frac{S^*+U}{S^*}}$$

It can be directly verified that the second-order derivative $\frac{\partial^2}{\partial\sigma^2}V_0^{(3)}$ is negative at $\sigma = \widehat{\sigma}$. So, we can state that

$$\widehat{\sigma} = \arg\max_{\sigma} V_0^{(3)}(\sigma)$$

As we can see the optimal insurer's risk depends on the FD barrier D, the initial value S_0 of company's surplus, the regulatory capital S^*, the parameter U of FD costs and the length T of the period between two consecutive audits. Taking the first-order derivative of $\widehat{\sigma}^2$ w.r.t. the FD D gives

$$\frac{\partial}{\partial D}\widehat{\sigma}^2 = \frac{2}{D\,T\ln\frac{S^*+U}{S^*}}\left(\ln\frac{D^2}{S^*(S^*+U)} + \ln\frac{D^2S^*}{S_0^2(S^*+U)}\right)$$

we conclude that $\frac{\partial}{\partial D}\widehat{\sigma}^2 < 0$ since according to the model setup $D < S^* \le S_0$. This means that when the FD barrier D increases, the company becomes more sensitive to the FD costs. Since such costs are caused by assumed risk, the company's risk managers will then tend to reduce such risk. It is obvious that the optimal company's risk is inversely proportional to the length T of the observed period. Taking the first-order derivative of $\widehat{\sigma}^2$ w.r.t. the company's initial level S_0 of capitalization

$$\frac{\partial}{\partial S_0}\,\widehat{\sigma}^2 = -\frac{2}{S_0\,T \ln\frac{S^*+U}{S^*}}\,\ln\frac{D^2}{S^*(S^*+U)}$$

we see that $\frac{\partial}{\partial S_0}\widehat{\sigma}^2 > 0$ since $D < S^*$. This means that the higher the company's initial level of capitalization the more risk it is willing to assume. And finally, more importantly, we consider the company's risk management sensitivity with respect to the parameter U of FD costs.

$$\begin{aligned}
\frac{\partial}{\partial U}\,\widehat{\sigma}^2 &= \frac{\partial}{\partial U}\,\left(\frac{1}{T}\,\frac{\ln\frac{S_0^2(S^*+U)}{D^2S^*}\,\ln\frac{S^*(S^*+U)}{D^2}}{\ln\frac{S^*+U}{S^*}}\right)\\
&= \frac{1}{T(S^*+U)}\,\frac{\left(\ln\frac{S_0^2(S^*+U)}{D^2S^*} + \ln\frac{S^*(S^*+U)}{D^2}\right)\ln\frac{S^*+U}{S^*} - \ln\frac{S_0^2(S^*+U)}{D^2S^*}\,\ln\frac{S^*(S^*+U)}{D^2}}{\ln^2\frac{S^*+U}{S^*}}\\
&= \frac{2\left(\ln\frac{S_0S^*}{D^2} + \ln\frac{(S^*+U)}{S^*}\right)\ln\frac{S^*+U}{S^*} - \left(2\ln\frac{S_0}{D} + \ln\frac{(S^*+U)}{S^*}\right)\left(2\ln\frac{S^*}{D} + \ln\frac{(S^*+U)}{S^*}\right)}{T(S^*+U)\,\ln^2\frac{S^*+U}{S^*}}\\
&= \frac{1}{T(S^*+U)}\,\frac{\ln^2\frac{S^*+U}{S^*} - 4\ln\frac{S_0}{D}\ln\frac{S^*}{D}}{\ln^2\frac{S^*+U}{S^*}}. \qquad (15)
\end{aligned}$$

From (15) we can see that $\frac{\partial}{\partial U}\,\widehat{\sigma}^2 > 0$ iff $U > U' = \left(\mathrm{e}^{2\sqrt{\ln\frac{S_0}{D}\,\ln\frac{S^*}{D}}} - 1\right)S^* > 0$. That is, when FD costs are extremely high (or the FD parameter U is close to the level of insolvency S^*) then the states *"insolvent"* and *"financially distressed"* are almost the same and this is the situation where it is too late to control risk (i.e. the company is nearly under liquidation). However, for reasonable values of FD costs, i.e. for all $U > U' > S^*$, we showed that the optimal company's risk decreases with increase of FD costs (or with decrease of the FD parameter U). We end this subsection by considering the question as to how insurance risk managers can adjust the value of risk σ to the optimal level $\widehat{\sigma}$. From our model of the company's surplus we can see that the integrated risk σ is

$$\sigma = \sqrt{\alpha^2\lambda^2\sigma_L^2 + (\alpha\lambda+1)^2\sigma_I^2}.$$

i.e. it depends on variable parameter of proportional reinsurance α and parameter of investment risk σ_I. Therefore, the company's risk managers can control the integrated risk $\sigma(\alpha,\sigma_I)$ in a two-dimensional way:

- control ex-post underwriting risk through reinsurance (by choosing α);
- control investment risk σ_I through risk management of its investment portfolio.

4 Conclusion

In this paper, we have investigated the risk management incentives in insurance companies in the presence of such particular type of frictional costs as financial distress costs. We considered modelling of insurer solvency in the presence of four different forms of financial distress costs, and in the case of RBC solvency model with financial distress costs in the form of reduced upside potential of the terminal value of insurer's surplus we demonstrated that there are risk management incentives to control underwriting and investment risks through reinsurance and investment hedge. We showed that these insurance risk management incentives increase with an increase of financial distress costs.

The results of this paper are very important and show that the decision to reinsure can be treated as both a risk-management and a capital-structure tool in shareholders' value creation.

References

[1] Andrade, G. and Kaplan, S. How costly is financial (not economic) distress? Evidence from highly levereged transactions that became distressed, *Journal of Finance*, Vol.53, pp.1443-1493, 1998.

[2] Briys, E. and de Varenne, F. Valuing risky fixed rate debt: an extension, *Journal of Financial and Quantitative Analysis*, Vol. 32, No 2 (June), pp. 239-248, 1997.

[3] Culp, C. *The ART of Risk Management*, John Wiley and Sons, 2002.

[4] Diamond, D.W. Debt maturity structure and liquidity risk, *Quaterly Journal of Econimics*, Vol. 106, pp. 709-737, 1991.

[5] Duffie, D. *Dynamic asset pricing theory*, Princeton University Press, 2001.

[6] Etheridge, A. *Financial Calculus*, Cambridge UP, Cambridge, 2002.

[7] Froot, K., Scharfstein, D. and Stein, J. Risk management: coordinating corporate investments and financing policies, *Journal of Finance*, Vol. 5, pp. 1629-1658, 1993.

[8] Geman, H., El Karoui and Rochet, C. Change of numéraire, changes of probability measure and option pricing, *Journal of Applied Probability*, Vol. 32, pp. 443-458, 1995.

[9] Jarrow, R. and Purnanandam, A. Capital structure and the present value of firm's investment opportunities: a reduced form credit risk perspective, *Working Paper, Johnson Graduate School of Management, Cornell University, NY*, January 15, 2004.

[10] Karatzas, I. and Shreve, S. *Brownian motion and stochastic calculus*, Springer, New York, 1988.

[11] Krvavych, Y. and Sherris, M. Enhancing insurer value through reinsurance optimization, *Insurance: Mathematics and Economics*, Vol. 38, pp. 495-517, 2006.

[12] Merton, R. and Perold, A. Theory of risk capital in financial firms. In J.M. Stern and D.H. Chew, editors, *The revolution in corporate finance.*, Malden, MA: Blackwell Business,Malden, MA, 1998.

[13] Musiela, M. and Rutkowski, M. *Martingale methods in financial modelling*, Springer, New York, 1998.

[14] Opler, T. and Titman, S. Financial distress and corporate performance, *Journal of Finance*, Vol.49, pp.1015-1040, 1994.

[15] Outreville, F. *Theory and Prctice of Insurance*, Kluwer AP, 1998.

[16] Powers, M. A theory of risk, return and solvency, *Insurance: Mathematics and Economics*, Vol. 17, pp. 101-118, 1995.

[17] Sherris, M. and van der Hoek, J. Capital Allocation in Insurance: Economic Capital and the Allocation of the Default Option Value, *North American Actuarial Journal*, April, 2006 (in press).