Mortality Fluctuations Modelling with a Shared Frailty Approach

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2nd December 2005

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Abstract

After a brief characterisation of the mortality fluctuations, we propose a shared frailty model for pricing and risk measurement in life insurance. The estimation of the parameters is computed on two models. The first model is a traditional Gompertz model, the second is a Gompertz model augmented with a frailty term representing the fluctuations. Reinsurance premia are then computed using the two models with their respective estimated parameters. We show that non-linear tarification and risk measures can be significantly increased when the frailty term is incorporated. We obtain qualified results, depending on other risk factors such as age and interest rates.

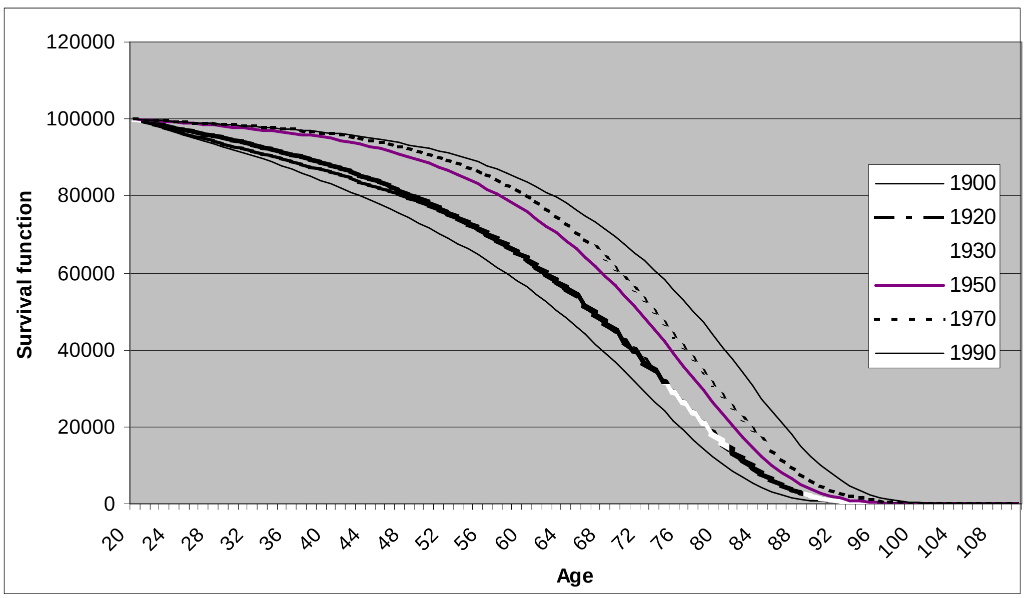
Key Words : Lifetimes Dependence, Frailty, Gompertz model, EM Algorithm, Term Life Insurance, Life Annuities, Excess of Loss Reinsurance Arrangement, Percentile Principle

JEL : G13

Contents

1 Introduction

During past decades, an increase in longevity has been observed. According to Olivieri and Pitacco [2002], the shape of the survival function in many developed countries evolved according to an expansion and a rectangularization pattern. Expansion corresponds to an increase of the maximum age of the survival curve towards very old ages. Rectangularisation means that the survival curve moves towards a rectangular aspect, due to more concentration of deaths at very old ages and less concentration at younger ages. We can observe these effects on a series of past survival functions, for successive periods from 1900 to 1990, excluding war periods :



French males survival curves for different periods, from 1900 to 1990 (Source : Human Mortality Database)

Similarly, Vaupel and Oeppen [2002] postulate that life expectancy was not close to a maximum and that early researchers who had asserted that life expectancy was approaching a ceiling, like Dublin [1928], have repeatedly been proven wrong.

Besides the increase in longevity, another phenomenon is observed. The mortality at young ages is not improving, and there is a larger dispersion due to accidental deaths at young ages (Pitacco [2002], Wang and Brown [1997]).

Moreover, the inclusion of war periods survival curves shows the importance of dispersion in the mortality of age group 25-50.

Impact on life insurance portfolios. Tosetti and al. [2000] have described the impact of an error on the choice of the mortality table and the interest rate on the pricing of life insurance contracts. It is analogous to study the impact of a mortality deviation on the result of life insurance portfolios. The authors conclude that impact of mortality deviations is comparable to impact of interest rate movements in a portfolio of life annuities. In the case of term life insurance contracts, mortality deviations deviations sensitivity is more important than interest rate sensitivity.

The case of life annuities is very important, since the number of such contracts in the insurance market is increasing significantly in several countries. This development in life annuities market appears as a natural consequence to the increase in longevity described previously. For instance, in France, the Plan d'Epargne Retraite Populaire (PERP) is most of the time (nearly 100%) converted into life annuities at retirement time. This adds longevity risk to be borne by the retirement system.

Development of life reinsurance and mortality risk titrisation. The *life reinsurance* market has significantly increased between 1990 and 2000. According to Clark [2003], only 15% of US life insurance business was reinsured in 1993. In 2000, this percentage grew up to 64%. Also, a concentration effect has to be noted. In 1995, 17 life reinsurers represented 90% of the life reinsurance market. These same 90% were only represented by 10 companies in 2002. In parallel to this consolidation phase, the development of the small companies, called "Bermudeans" has been observed.

The longevity risk titrisation is another mortality risk hedging solution being currently developed. For instance, BNP Paribas has structured a *Longevity Bond*, issued by the European Investment Bank, in association with PartnerRe for the longevity expertise. This bond aims at hedging the longevity risk borne by potential clients like insurance companies or pension funds. The Longevity Bond consists in annual coupons indexed to the observed cumulative survival rates of a cohort of individuals.

New regulatory framework : Solvency II. The European Commission issued in 2000 a proposal of insurance prudential regulation revision. This project, named "Solvency II", has closely followed a preliminary study (Solvency I, KPMG Study [2002]) aiming at observing the rules for calculating the solvency margin requirements. With Solvency II, regulators aim at a consistent risk management across european insurance companies and consistent across the financial sector globally. The case of a life insurance company, mortality risk and policyholder behavior risk are the two main components. Solvency II highlights the importance to represent the trends in mortality improvements, the mortality deviations and shocks leading to mortality deterioration (KPMG Study [2002]).

Approaches to mortality deviations modelling. The mortality deviations risk can be modelled by several different approaches. Some of those approaches put more emphasis on the trend in longevity increase (Lee-Carter [1992]), whereas some other merely focus on the dispersion aspect of the mortality deviations (Pitacco [2002]). A particular approach, the shared frailty model, has already been used by demographs (Yashin [1995]) in order to represent the mortality fluctuations. In this model, an external random factor, has a common action on the mortality of a group of individuals. This situation can be illustrated by a group of hemophils, where the sensitivity of these individuals to medical improvement (the external common unobserved risk factor) creates extra variance in their mortality risk. The frailty model will be studied in this paper. Our aim is to analyse the impact of incorporating a mortality fluctuations component into a traditional model, on life insurance portfolios.

Plan of the article. First, in order to justify and position our approach in the cartography of existing models, we will briefly outline some of the most known techniques to mortality deviations modelling in the next section. In section 3, we will introduce the shared frailty model in more detail, and make the specification of its underlying functions and distributions. Results of the estimation of a model without frailty and the shared frailty model are presented. We use two data sets with low and high dependence respectively. Finally, in section 4, we will study the application of the frailty model to life reinsurance pricing. We show that the integration of dependence increases the non-linear life reinsurance premia, when compared to the results obtained in a traditional model.

2 Mortality models

In this section, we will caracterise the mortality deviations and give a brief overview of some existing models. The models can be split in two categories. The first category concerns the modelling of mortality trends, such as the increase in longevity. Such models are relevant for pricing life annuities over a long term. The second approach focuses on the mortality risk dispersion or fluctuations, and would be more relevant to term life insurance portfolios.

2.1 Causes of mortality deviations

Model error. Model error can be one of the causes of mortality deviations. Model error occur when the wrong mortality table has been used for the pricing or when the probabilistic model has been mis-specified. For instance, one may have used a non-smoker mortality table for reserving life insurance contracts of smokers.

Another possible cause of mortality deviations is an insufficient size of the population, so that the law of large numbers cannot be verified. Small population size generates the random deviations by opposition to systematic deviations (Olivieri [2002]). Similarly, a portfolio containing several contracts with small benefits and a few with very large benefits, may also lead to mortality deviations from an insurer's perspective.

When the portfolio is large and homogeneous, the mortality deviations are of systematic nature. They can be generated by the common action of some unobservable risk factors on the mortality of the population. These risk factors generate dependence of lifetimes (Hougaard [2000]). We can split the lifetime dependence mechanisms into two categories as follows.

Common events dependence. This type of dependence is defined as a single cause leading to several simultaneous deaths. A common event leads to very strong dependence during a very short period of time. Examples of common events can be plane crashes, explosion of a chemical factory such as AZF Toulouse in 2001, Terrorism (WTC attacks in 11 September 2001), natural catastrophes (Asian Tsunami in December 2004 causing hundreds of thousands of deaths, or the earthquake in Iran causing around 40000 deaths), mass epidemics of explosive nature (Ebola Virus) where the virus is so lethal that it can only spread over for a short time.

A characteristic of such models is the singularity of the multivariate survival function $(t_1, ..., t_n) \longrightarrow \mathbb{P}(\tau_1 > t_1, ..., \tau_n > t_n)$ on the diagonal line $t_1 = t_2 = ... = t_n$ since they allow for several simultaneous deaths. A well known example of common events model is the exponential shock model of Marshall and Olkin [1967].

Common risks dependence. According to Hougaard [2000], lifetime dependence is generated when there are some *unobservable risk factors*, acting on the mortality of *several* individuals in a *common* manner. This dependence is called "*common risks dependence*". The level of dependence between individual lifetimes is directly related to the variance of the unobservable common risk factors. If the variance of those factors is very high, the dependence will be strong. As a corollary, the common risks dependence has the *conditional independence* property : by conditioning on the unobservable common risk factors, the lifetimes of the individuals are independent.

Examples of common risks could be : the mutation of viruses and bacterias (Spanish Flu in 1918, SARS in Asia in 2003), the silent development of a new epidemics (AIDS), the unknown action of a new pollution (Asbestos) or the medical and public health improvement leading to an increase in longevity.

As suggested by these examples, common risk factors are the consequence of numerous and fast changes in life practices. For instance, development of transports and communications could lead to the threat of new epidemics worldwide, like the SARS in 2003. Even if it is generally accepted that pollution levels are decreasing, epidemiologists are sceptical about the use of substitutes and their potential effect on the mortality of the population as a whole.

2.2 Intensity models

These models assume a dynamic for the mortality rate, also called the intensity. In this framework, the individuals are assumed independent. Consequently, the individual mortality rate processes are independent. We will give a general formulation, proposed by Devolder [2005], then derive some particular cases well known in actuarial practice and demography.

General formulation. For a given individual with age x at year t, the mortality rate $\mu_x(t)$ can be expressed as the solution of the stochastic equation :

$$d\mu_x(t) = \nu(x, t, \mu)\, dt + \sigma(x, t, \mu)\, dW_t$$

where $\{W_t\}_t$ is a Brownian motion. This is an individual model, with all individuals lifetimes and mortality rates assumed independent. We will now give some particular cases.

Devolder Approach. Devolder [2005] proposed to build an affine continuous process by taking :

$$\begin{aligned} \nu(x, t, \mu) &= \nu(x, t)\, \mu_x(t) + \beta(x, t) \\ \sigma(x, t, \mu) &= \sqrt{\sigma(x, t)\, \mu_x(t) + \xi(x, t)} \end{aligned}$$

The advantage of this formulation is the mathematical tractability, leading to the Ricatti equations. It is also possible to introduce some features on the dynamic of the mortality rate, such as a mean reversion towards an expected limit mortality rate $\widetilde{\mu}_{x+t}$, or a biological absolute limit mortality rate μ^*_{x+t}.

Gompertz model. This is the traditional model, used for the standard mortality tables. We find this model when we take $\nu(x,t,\mu) = cte$. Solving the equation and using traditional notations, we end up with :

$$\mu_x(t) = \gamma \exp[p(t+x)] = \exp(\alpha_{x+t})$$

Lee-Carter method. This method is currently considered as a reference by many actuaries and demographs. It can be obtained when taking :

$$\begin{aligned} \nu(x,t,\mu) &= \left(\alpha'_{x+t} + \beta_x(t) + t\beta'_x(t)\right)\mu_x(t) \\ \sigma(x,t,\mu) &= \sigma\mu_x(t) \end{aligned}$$

We have a log-normal process for $\mu_x(t)$ with a deterministic drift and a constant volatility (although this may not be an obligation, but we keep it for the sake of simplicity). Applying Itô's Lemma to $\ln\mu_x(t)$ then proceeding to integration leads to

$$\mu_x(t) = \exp\left\{\alpha_{x+t} + \left(\beta_x(t) - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}$$

By changing the notations, we get the initial formulation of the Lee-Carter model [1992], where the idea is to split the mortality rate $\mu_x(t)$ at age x in year t into an age specific component and a time specific component :

$$\mu_x(t) = \exp(\alpha_x + \beta_x\kappa_t + \varepsilon_{xt})$$

In order to ensure identifiability of the model, the constraints $\sum_t \kappa_t = 0$ and $\sum_x \beta_x = 0$ have been introduced. Most of the variance of $\mu_x(t)$ at each age x is explained by κ_t, so ε_{xt} is only a white noise. The aim of the method is to make future projections from past data analysis. For instance, Denuit and Brouhns [2002] have applied the method to data made of the numbers of deaths $d_x(t)$ at age x and year t, with $x \in \{60, ..., 98\}$ and $t \in \{1960, ..., 1998\}$. This model is the most relevant for mortality projections over a long period of time. It aims at representing a trend in mortality rates movements, using past mortality data, in order to provide mortality rate at a given age in a future date.

According to Vaupel [2002], there are some drawbacks to the Lee-Carter method. Firstly, it assumes that the death rates will decline at the same average pace as they have done in the past. Secondly, the confidence bands for future life expectancy are based on variance of past death rates. Therefore, the method doesn't take into account the extra dispersion that could be created by an unexpected impact of medical progress on the life expectancy. In fact, the model doesn't include any assumption on medical improvement evolution, on the possibility of new diseases or on the change in life practice. This makes the Lee-Carter model a relevant approach for the life annuities but not for the life insurance portfolios, where a proper representation of mortality dispersion is very important.

Extensions of Lee-Carter method. Other approaches derived from Lee-Carter approach include Lee [2000], Renshaw and Haberman [2001]. Brouhns and Denuit [2001] have suggested to adjust a log-bilinear Poisson model. The number of deaths $d_x(t)$ is assumed to follow a Poisson process defined as :

$$d_x(t) \sim Poisson\left(L_x(t)\,\mu_x(t)\right)$$

with $\mu_x(t) = \exp\left(\alpha_x + \beta_x \kappa_t\right)$ and $L_x(t)$ is the number of x aged individuals at year t. The advantage of this parametric model is the possibility to perform maximum likelihood estimation of the parameters, instead of the least squares estimates used in the traditional Lee-Carter method.

2.3 Models of lifetimes dependence

The intensity models previously described assume independence of the individual lifetimes. As a consequence of the law of large numbers, the variance can be reduced up to zero when taking a large and homogeneous portfolio. Vaupel [2002] mentioned that the projection methods do not integrate the extra-variance due to some mortality flucutations. In the subsection (2.1), we suggested that mortality fluctuations could be represented by models of lifetime dependence. In the present subsection, we will briefly outline some of the main lifetime dependence models.

Comonotonicity. This is a case of perfect dependence. A set of random variables $(Y_1, ..., Y_n)$ is comonotonic if there exists a random variable Z and a set of non-decreasing functions $(g_1, ..., g_n)$ of the real variable such that we have the equality in distribution

$$(Y_1, ..., Y_n) \sim \left(g_1(Z), ..., g_n(Z)\right)$$

Such a model has been applied to life insurance by Dhaene and Goovaerts [1997]. They have shown that stop-loss premiums increased by significant amounts (up to 300%). Due to this rather extreme aspect, the comonotonicity can be considered as a conservative approach. Ribas, Marin-Solano and Alegre [2002] suggest the use of more intermediate forms of dependence in the case of lifetime dependence. In reality, dependence of lifetimes should be closer to independence than to perfect correlation.

Mixture models. This is a Bayesian approach. For instance, this approach was suggested by Pitacco and Olivieri [2002] for an application to life insurance. They assume that the lifetimes $\tau_1, ..., \tau_n$ follow a $Weibull\,(\alpha, \beta)$ distribution, where the parameters α and β are random. Conditionally on α and β, the lifetimes $\tau_1, ..., \tau_n$ are independent. According to the authors, the expansion and rectangularization phenomena can be measured with the mean, the mode and the variance of the lifetimes. These moments can be easily derived in the Weibull distribution.

The shared frailty model. This model has already been used by demographs in order to model lifetime dependence of twins or individuals of a married couple (Yashin [1995]). As suggested by Hougaard [2000], the shared frailty model represents the lifetime dependence generated by the action of a set of external common risk factors. Applications of the frailty model to life insurance have been proposed (Valdez [1999], Haberman and Butt [2002]), however, these applications only concern individual frailty and the representation of the heterogeneity of mortality.

Wang and Brown [1998] used the frailty model in order to represent the increase in longevity. They have mentioned one advantage of the frailty approach : unlike most of the mortality projection models, the frailty model is based on the underlying biological mechanism of mortality improvement. In their model, they postulate that the more frail people make a better profit of medical progress, thus increasing their longevity by a bigger amount than the more resistant people. The shared frailty model will be described in more details in the next section.

3 Formulation of the shared frailty model

In this section, we will explain the assumptions and properties of the shared frailty model as well as the specification of the terms of the model. Finally, we will propose an estimation of the parameters.

3.1 Assumptions and notations

The shared frailty model has been suggested by Hougaard [1984] for the modelling lifetimes dependencies within a group of individuals. In the model, each individual has a mortality rate function of the observed risk factors such as age or sex (like in a standard Cox model [1972], or a Gompertz model). In addition, an unobservable risk factor, the *frailty*, operates on the mortality of all the individuals in a common manner, generating common risk dependence. For the sake of simplicity, we assume there is only one group of individual sharing the same frailty. We start with giving some notations :

- Z is the frailty shared by all individuals of the group
- $\varphi(z)$ or $\varphi(z;\boldsymbol{\theta})$ is the common density probability of frailty Z, $\boldsymbol{\theta}$ being the vector of parameters of the frailty distribution.
- $\Psi(s)$ or $\Psi(z;\boldsymbol{\theta})$ is the Laplace transform of the distribution of the Z, by definition equal to $\mathbf{E}\left(e^{-sZ} \mid \boldsymbol{\theta}\right)$ for all $s \in \mathbb{R}_+$.
- $\mu_j(t \mid Z)$ is the mortality rate of individual j, conditional to Z
- $\mu_{0j}(t)$ is the baseline hazard rate of individual j, and $M_{0j}(t)$ is the integrated hazard rate

The assumptions of the shared version of the frailty model are the following :

Assumption (A1) *conditional on the frailty Z, the random lifetimes of individuals within the group are independent*

Assumption (A2) *the frailty Z has a multplicative effect on the mortality rate of the individuals :*

$$\mu_j(t \mid Z) = Z\mu_{0j}(t) \tag{1}$$

Assumption (A3) *the frailty Z is stationnary*

The dependence of lifetimes within the group is generated by the variance of the frailty. The greater Z, the higher the risk of dying from a common cause in the group. On the other side, assumption (A1) tells that independence of individuals is obtained when Z is known, which is equivalent to $\mathbf{Var}(Z) = 0$.

3.2 Specification of the elements

The dependence structure is obtained through the distribution of the frailty Z. We need to restrict our choice to distributions with a strictly positive support, since negative frailty leads to negative mortality rates and means immortal individuals. Gamma distribution, Inverse-Gaussian, positive stable distribution or power variance functions can be assumed. The dependence structure enhanced by those distributions are explained in Hougaard [2000] and can be considered when modelling the loss distribution of an insurance portfolio. For instance, if the insurer wants to model extreme events, he should consider distributions with a stronger tail dependence, like the positive stable distribution.

The baseline individual hazard rate will follow the Gompertz law. We choose to include no covariate apart from the age x_j at the entry date. In this case, the individual baseline hazard rate has the following expression :

$$\mu_{0j}(t) = \gamma \exp\left[p\left(t + x_j\right)\right]$$

where γ and p are the Gompertz parameters. For more details on the characteristics of the Gompertz distribution we refer to Pollard and Volkovics [1993].

For the frailty random variable, we will use the Gamma distribution. The Gamma distributed shared frailty model is easily tractable and has already been used in many other fields, such as demography or epidemiology (Yashin [1995], Hougaard [2000]). Since we will refer to the parameters of the rest of the paper, we have found usefull to recall the density function :

$$\varphi(z) = \frac{\theta^{\delta} z^{\delta-1} \exp(-\theta z)}{\Gamma(\delta)}$$

with $\delta > 0$ et $\theta > 0$. For identifiability, it is usually assumed $\mathbf{E}(Z) = 1$ (Klein [1992]) so we have the following equality :

$$\theta = \delta$$

3.3 Probabilistic functions

We will now describe the basic probabilistic functions of the shared frailty model. We focus our interest essentially on the individual and the multivariate survival functions, as they are used several times, either in the parameters estimation process or in the derivation of the portfolio loss distribution.

- **Individual survival function, conditional to frailty**

$$\mathbf{P}\left(\tau_j > t \mid Z\right) = \exp\left(-Z M_{0j}\left(t\right)\right) \tag{2}$$

- **Multivariate survival function, conditional to frailty**

Using above expression, and conditional independence assumption (A1), one can write the conditional multivariate survival function of lifetimes $\tau_1, ..., \tau_n$ as :

$$S\left(t_1, ..., t_n \mid Z\right) = \exp\left[-Z\left\{\sum_{j=1}^{n} M_{0j}\left(t_j\right)\right\}\right] \tag{3}$$

- **Individual survival function**

We obtain this function after taking the expectation with respect to Z of the conditional individual survival function :

$$S_j\left(t\right) = \Psi\left(M_{0j}\left(t\right)\right) \tag{4}$$

- **Multivariate survival function**

Similarly, we calculate the multivariate survival function as a function of the Laplace transform Ψ and the individual integrated mortality rates :

$$S\left(t_1, ..., t_n\right) = \Psi\left(\sum_{j=1}^{n} M_{0j}\left(t_j\right)\right) \tag{5}$$

It is interesting to note that if the Laplace transform is invertible, the multivariate survival function can also be expressed in terms of the individual survival functions only. This establishes the equivalence of the frailty model with the Copulae (Joe [1997], Frees and Valdez [1998])

- **Multivariate density function**

This function is obtained after derivation with respect to each $t_1, ..., t_n$ of the multivariate survival function :

$$(-1)^n \left\{ \prod_{j=1}^{n} M_{0j}\left(t_j\right) \right\} \Psi^{(n)} \left(\sum_{j=1}^{n} M_{0j}\left(t_j\right) \right)$$

3.4 Estimation of the parameters

The aim of this paragraph is to provide some intuition of the possible values of the dependence parameters, in order to use realistic parameters in the application to life insurance. Therefore, we will not go into much detail of the estimation algorithm.

The data have been downloaded from the Human Mortality Database, a website co-founded by the Institut National des Etudes Démographiques, the Max Planck Institute for Demographic Studies and the University of Los Angeles. We observe french populations from 1900 to 1997. The groups are made of males, aged from 30 to 80 years at the beginning of the study, and being observed for 5 years. Note that this is similar to the approach used in Frees, Carriere, Valdez [1996]. Each group corresponds to a period of observation of 5 years length : 1900-1905, 1905-1910, ... , 1990-1995.

- Data Set 1: we have excluded the war periods. This data set represents a population with low dependence a priori

- Data Set 2 : we have included the war periods. This data set represents a population with high dependence a priori

The use of these two data sets will give an idea of the range of possible values of the frailty parameter (δ). Later on, it will help to quantify the impact of taking the dependence into account in the risk management of life insurance portfolios.

The resulting estimators have been calculated for the Gamma-Gompertz model, using the EM algorithm. This is an iterative bayesian method, aiming at maximizing the likelihood in case of incomplete data. It is therefore relevant to the frailty model, since the random effect is not directly observed. We refer to Klein and Moeschberger [1997] for a description of the EM algorithm in the frailty case and we refer to Dempster, Laird and Rubin [1977] for a theoretical justification of the EM algorithm. The estimated parameters are given in the two tables below, for data set 1 and 2 respectively.

Data Set 1	**Gompertz standard**	**Gamma-Gompert Frailty**
$\widehat{\gamma}_1$	$\widehat{\gamma}_1^{Indep} = 0.000122$	$\widehat{\gamma}_1^{Frailty} = 0.000116$
$\widehat{p}_1$	$\widehat{p}_1^{Indep} = 0.081016$	$\widehat{p}_1^{Frailty} = 0.081826$
$\widehat{\delta}_1$	-	$\widehat{\delta}_1^{Frailty} = 227.6571$
$\widehat{\theta}_1 = \widehat{\delta}_1$	-	$\widehat{\theta}_1^{Frailty} = 227.6571$

Data Set 2	**Gompertz standard**	**Gamma-Gompert Frailty**
$\widehat{\gamma}_2$	$\widehat{\gamma}_2^{Indep} = 0.000227$	$\widehat{\gamma}_2^{Frailty} = 0.000231$
$\widehat{p}_2$	$\widehat{p}_2^{Indep} = 0.077562$	$\widehat{p}_2^{Frailty} = 0.078801$
$\widehat{\delta}_2$	-	$\widehat{\delta}_2^{Frailty} = 10.06357$
$\widehat{\theta}_2 = \widehat{\delta}_2$	-	$\widehat{\theta}_2^{Frailty} = 10.06357$

We observe that the parameter δ is around 10 for a strong dependence and around 200 for a low dependence. Moreover, the Gompertz parameters γ and p are not significantly changed by the introduction of a frailty. In fact, they only adjust in order to conserve the value of the marginal mortality rate, this one being an exponential function of γ and p. Finally, we can compare the values of the parameters to the results of Haberman and Butt [2002]. These authors use the frailty for the representation of heterogeneity in individual mortality. However, their results give an idea of possible values for the Gamma-Gompertz parameters. As expected, our δ is higher, because the individual mortality is less driven by common unobservable risk factors than by individual unobservable risk factors.

3.5 Measure of the dependence obtained

As explained in previous subsections, the lifetime dependence in the frailty model is generated by the variability of an external common unobservable risk factor. Therefore, it makes sense to use the coefficient of variability of the frailty

$$CV^2(Z) = \frac{\mathbf{Var}\,(Z)}{\mathbf{E}\,(Z)}$$

in order to measure the dependence (Hougaard [2000]). Some other measures of dependence include the traditional Pearson linear correlation coefficient, the coefficient of concordance or Kendall's tau, the median concordance and Spearman's correlation coefficient (see Embrechts, McNeil and Strautmann [1998] for a detailled study of dependence measures). In Figure 1, we observe that a Kendall Tau coefficient (green curve) or a Spearman Rho (pink curve) are close to the $0.5CV^2(Z)$ (blue curve). We can reasonnably assume, in the case of the Gamma-Gompertz model, that $CV^2(Z)$ is consistent with Kendall's Tau and Spearman Rho correlation coefficient. We will use $CV^2(Z)$ as a measure of dependence later in the article.

Now taking the unit expectation assumption $E\,(Z) = 1$ into account, for $\delta > 0$, we obtain :

$$CV^2(Z) = 1/\delta$$

and we calculate the dependence levels $CV^2(Z) = 0.44\%$ for the Data Set 1 and $CV^2(Z) = 9.94\%$ for the Data Set 2.

4 Application to life insurance

In the previous sections, we have described our model. From a standard Gompertz model we introduced a common frailty term representing the dependence, leading to the Gamma-Gompertz shared frailty model. We have estimated the parameters of Gompertz model and Gamma-Gompertz model on 2 data sets, one with low dependence a priori, the other with high dependence a priori.

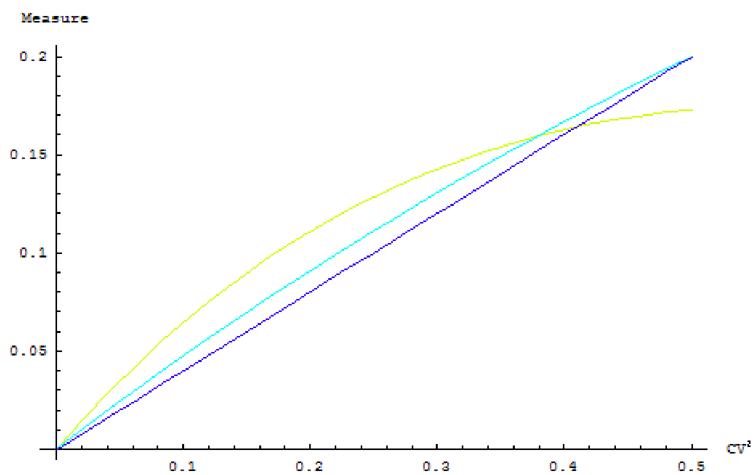


Figure 1: *Comparison of $CV^2(Z)$ with Kendall coefficient of concordance and Spearman correlation coefficient.*

4.1 Obtention of the portfolio aggregate loss distribution

4.1.1 Notations and formulation

General case. The individual model states that the aggregate risk Y of a portfolio is the sum of the individual risks Y_j contingent to an event. The event related to individual risk Y_j occurs at a random time τ_j. We insist that the portfolio is not necessarily homogeneous in terms of individual risks and benefits size. This portfolio is not closed either. We will note $Y_j(t)$ the deterministic present value of the individual risk if $\tau_j = t$. We have

$$Y = \sum_{j=1}^{n} Y_j$$

and we can represent the individual risk as

$$Y_j = \int Y_j(t) \mathbf{1}_{\{\tau_j = t\}} dt$$

Case of a Life insurance portfolio. We can use some of the formalism introduced by Petauton [2001]. However we have restricted this formalism to the discrete case and assumed the benefit payments made at the end of the year of death in case of life insurance contracts :

- Y_j is the random present value (at $t = 0$) of the liability of insurer for the contract j
- Y is the random present value (at $t = 0$) of the total liability of insurer
- $B_{j,k}$ is the deterministic present value (at $t = 0$) of the payment made by the insurer if $\tau_j \in [k-1, k[$
- $C_{j,k}$ are the death benefits, paid at date k if $\tau_j \in [k-1, k[$
- $c_{j,k}$ are the life benefits, paid at date k if the annuitant j is alive at date k
- $\pi_{j,k}$ are the pure annual premiums, to be received at date k by the insurer if the annuitant j is still alive at k
- $v(0, k)$ is the discount factor from 0 to k

For a general insurance contract we can write :

$$B_{j,k} = C_{j,k} v(0, k) + \sum_{r=0}^{k-1} (c_{j,r} - \pi_{j,r}) v(0, r)$$

The random variable Y_j can be expressed in terms of payoff, depending only on the realisation of random lifetime τ_j :

$$Y_j = \sum_{k=1}^{T_j} B_{j,k} \mathbf{1}_{\{\tau_j \in [k-1,k[\}} + B_{j,T_j} \mathbf{1}_{\{\tau_j > T_j\}}$$

Case of a life reinsurance portfolio. We use analogous notations than in the case of a life insurance portfolio :

- $\widetilde{Y}_j$ is the random present value of the liability of reinsurer related to the contract j
- $\widetilde{Y}$ is the random present value of the total liability of the reinsurer

- $\widetilde{B}_{j,k}$ is the deterministic present value of the payment made by the reinsurer, related to the j-th contract, if $\tau_j \in [k-1, k[$

Convergence for large homogeneous portfolios. It is possible to easily obtain the loss distribution for a large and homogeneous portfolio. This is the result of the application of the law of large numbers in the frailty context. In this case, the remaining risk driving the loss random variable is the frailty Z. De Finetti has proved that :

$$\frac{Y}{n} \longrightarrow \mathbf{E}\left(Y_1 \mid Z\right)$$

in distribution and almost surely, when $n \longrightarrow +\infty$.

4.1.2 Integration of the frailty term : PGF and FFT inversion

The frailty model leads to semi-analytical formulas for the portfolio loss distribution in the general case. The loss distribution can derived using the Fourier transform inversion of the probability generating function (PGF). For instance Rolski, Schmidli, Schmidt et Teugels [1999] have described these methods in an insurance context. The main difference in the present article is the dependence of lifetimes. The derivation is based on an application of the Levy-Khinchin theorem and can be computed in the three following steps :

- Calculation of the expectation of the individual risks, conditional on frailty

$$\mathbf{E}\left(u^{Y_j} \mid Z\right) = \int_0^{T_j} \mu_j\left(t \mid Z\right)\ S_j\left(t \mid Z\right)\ u^{Y_j(t)}\ dt$$

- Derivation of the probability generating function (PGF) of the aggregate risk, conditional on frailty, then obtention of the expectation with respect to Z using the conditional independence property of the frailty

$$\Phi_Y\left(u\right) = \mathbf{E}^Z\left[\prod_{j=1}^{n} \mathbf{E}\left(u^{Y_j} \mid Z\right)\right]$$

- Use the Fast Fourier Transform (FFT) in order to get the loss distribution, considering that we have the identity

$$\Phi_Y(u) = \mathbf{E}\left(u^Y\right) = \sum_l \mathbf{P}(Y = l)\ u^l$$

4.2 Application to an Excess of Loss (XL) reinsurance arrangement

4.2.1 Description of the XL contracts

An Excess of Loss (XL) contract in life insurance aim at covering the longevity risk of life annuity portfolios. In an XL arrangement, for each contract j the reinsurer pays the final part of the life annuity while exceeding a given term. We denote by h the maximum period of annuity payment from the cedant to each individual, hence if the annuitant j is alive at age $x_j + h$, the reinsurer starts paying the annuity, upon death of the annuitant. We assume that the cedant and the reinsurer agree upon the representation of the mortality models and parameters. Hence they adopt the same term structure of interest rates and same individual mortality risk distributions (Olivieri [2002]).

Assumption 1 *For the XL contracts, we assume that the cedant and the reinsurer use the same model and parameters for the mortality.*

This means that a given survival probability would be the same calculated by the insurer and the reinsurer. We can use this assumption without loss of generality. The aim of the assumption is to simplify the notations and calculations. Then, using the previous notations, assuming a unique pure premium at initiation for each annuity, we can write :

$$Y_j = \sum_{k=1}^{h} c_{j,k}\ \mathbf{1}_{\{\tau_j > k\}}\ v(0,k)$$

The reinsurer is generally left with two options. The first possibility is to pay the remaining liability on each annuity contract still in force as a lump sum to the cedant. Otherwise, he can choose to pay the annuities every year until each annuitant dies. We will assume this second option for our further study. Thus, the random present value of the future liability of the reinsurer on contract j can take the form :

$$\widetilde{Y}_j = \sum_{k=h+1}^{\omega-x_j} c_{j,k}\ \mathbf{1}_{\{\tau_j>k\}}\ v(0,k)$$

As proposed by Olivieri [2002], we choose to price the XL contract premium on the aggregate annuity portfolio of cession using the percentile principle. This is a non linear pricing method where premium is, for a given confidence level α :

$$q_\alpha\left(\widetilde{Y}\right) = \inf\left\{y|\mathbf{P}\left(\widetilde{Y} < y\right) \geq 1-\alpha\right\}$$

The life annuity contracts, components of the reinsurance portfolio, will have the following generic characteristics :

- The number of annuitants is $n = 10000$
- The term of the XL reinsurance contact is $h = 20$
- The age at annuity initiation is the same for all annuitant : $\forall j,\ x_j = x$
- Each annuity contract pays an amount of $e1000$ per annum, paid at the end of each year if the annuitant is alive
- The technical interest rate is $i = 3.5\%$
- The percentile level is $\alpha = 0.05$

4.2.2 Impact of the dependence on XL premia

We start with the pricing of the XL arrangement under the two set of estimated parameters found in the previous section, and simultaneously using the Gompertz (independence) and Gamma-Gompertz shared frailty model with the relevant set of parameters previously estimated, under each Data Set. To summarize :

- $Gompertz\left(\widehat{p}_1^{Indep}, \widehat{\gamma}_1^{Indep}\right)$ for Gompertz model calibrated on Data Set 1
- $Gompertz\left(\widehat{p}_2^{Indep}, \widehat{\gamma}_2^{Indep}\right)$ for Gompertz model calibrated on of Data Set 2

- *Gamma-Gompertz* $\left(\widehat{p}_1^{Frailty}, \widehat{\gamma}_1^{Frailty}, \widehat{\delta}_1^{Frailty}\right)$ for Gamma-Gompertz frailty model calibrated on Data Set 1
- *Gamma-Gompertz* $\left(\widehat{p}_2^{Frailty}, \widehat{\gamma}_2^{Frailty}, \widehat{\delta}_2^{Frailty}\right)$ for Gamma-Gompertz frailty model calibrated on Data Set 2

The numerical results are expressed in k€ ($\times e1000$) in the following tables
:

Data set 1 (low dependence population) : XL premia and dependence / independence ratios

Initiation Age	Gompertz	Frailty	Ratio
$x = 50$	$31,577$	$34,322$	1.09
$x = 55$	$21,564$	$23,969$	1.11
$x = 60$	$13,083$	$14,984$	1.15
$x = 65$	$6,730$	$8,028$	1.19
$x = 70$	$2,741$	$3,457$	1.26
$x = 75$	798	$1,087$	1.36

Data set 2 (high dependence population) : XL premia and dependence / independence ratios

Initiation Age	Gompertz	Frailty	Ratio
$x = 50$	$22,157.74$	$35,023.34$	1.58
$x = 55$	$13,786.36$	$24,833.91$	1.80
$x = 60$	$7,391.84$	$15,905.19$	2.15
$x = 65$	$3,220.91$	$8,859.01$	2.75
$x = 70$	$1,046.78$	$4,059.99$	3.87
$x = 75$	223.576	$1,411.56$	6.31

Firstly, the integration of a frailty led to an increase of the XL premia. This increase is significant, even if the dependence is low. For the Data Set 1, corresponding to a dependence level of 0.44% as measured by $CV^2(Z)$, we have found an increase of +15% for the initiation age $x = 60$. The increase due to dependence ranged from +9% to +36%.

Then, the dependence / independence ratio is increasing with age. In fact, this proves that the common risk factors have more impact on marginally riskier individuals. This underlines the selection property of the frailty model (Wang and Brown [1998]) : the more frail individuals will benefit the most from medical improvement.

Finally, we observe an increase of the XL premia between the two data sets, in the case of the frailty model. This is not happening in the independence case, where the premia calculated from the parametrisation of the Data Set 2 are lower than the premia calculated from the Data Set 1.

Remark. There is a very strong increase obtained in the second set of estimated parameters (Data Set 2), ranging from +58% to +731%. The second data set is similar to the first data set, but includes the war periods. There was not a so significant difference between the two data sets in terms of longevity improvement. The main difference was a greater dispersion due the war periods. This important increase highlights that the Shared Frailty model does not make a clear difference between mortality risk dispersion and longevity risk.

4.2.3 Effect of the XL reinsurance term

In this paragraph, we study the effect of the reinsurance term h, on the impact of dependence on the XL reinsurance premia $q_\alpha\left(\widetilde{Y}\right)$. It seems that the significant increase is due to the tail risk characteristic of the XL reinsurer aggregate loss $\widetilde{Y}$. The tail risk is not well represented by the independence model, thus leading to very small XL premium, especially in the high age categories. We can analyse the impact of the dependence on the distribution, when increasing the term h of the XL arrangement.

Data set 1, ratio $\frac{q_\alpha^{Frailty}\left(\widetilde{Y}\right)}{q_\alpha^{Indep}\left(\widetilde{Y}\right)}$ for different values of the term h

	$h=0$	$h=5$	$h=10$	$h=20$
$x=50$	1.027	1.036	1.049	1.09
$x=55$	1.031	1.043	1.059	1.11
$x=60$	1.036	1.052	1.074	1.15
$x=65$	1.041	1.063	1.092	1.19
$x=70$	1.047	1.076	1.117	1.26
$x=75$	1.053	1.093	1.150	1.36

Data set 2, ratio $\frac{q_\alpha^{Frailty}\left(\widetilde{Y}\right)}{q_\alpha^{Indep}\left(\widetilde{Y}\right)}$ for different values of the term h

	$h=0$	$h=5$	$h=10$	$h=20$
$x=50$	1.15	1.21	1.29	1.58
$x=55$	1.18	1.26	1.37	1.80
$x=60$	1.21	1.32	1.48	2.15
$x=65$	1.24	1.39	1.63	2.75
$x=70$	1.28	1.50	1.86	3.87
$x=75$	1.32	1.64	2.23	6.31

The case $h=0$ corresponds to the capital allocation for a portfolio of annuities without the XL reinsurance agreement. In this case, we observe already significant impacts of the dependence, ranging from +2.7% to +5.3% for the low dependence case, and from +15% to +32% in the high dependence case. The higher h, the higher the ratio of increase due to dependence. In fact, if h is high and the age x is high as well, we end up with an XL premium close to zero with the standard Gompertz model, whereas the frailty model gives some value to such contracts.

To investigate further the effects of the reinsurance term, Figure 2 displays curves of ratios dependence / independence $\frac{q_\alpha^{Frailty}(\widetilde{Y})}{q_\alpha^{Indep}(\widetilde{Y})}$ as a function of h for several values of $CV^2(Z)$: $0.5\%, 1\%, 2\%, 10\%$ and 20%. The The XL reinsurance term h magnifies the impact of the dependence in a convex manner when increased.

4.2.4 Effect of the interest rate

We also found worth explaining the relation between effect of interest rate and effect of dependence. Initially, we claimed that portfolios of annuities had similar sensitivity to interest rate and mortality assumption, so we want to validate this assertion. The Figure 3 the ratio of XL premia $\frac{q_\alpha^{Frailty}(\widetilde{Y})}{q_\alpha^{Indep}(\widetilde{Y})}$ as a function of interest rate i, for the Data Set 1 (low dependence) and Data Set 2 (high dependence).

We observe that the Data Set 2 curve displays a significant decrease with increasing interest rate i. The Data Set 1 curve has a very weak sensitivity to interest rate, staying around 1.2, with only a slightly, almost linear decrease, when i moves from 1% to 20%.

In both cases, the impact of dependence is higher in a low interest rate environment. This tells how good at representing the longevity risk is the

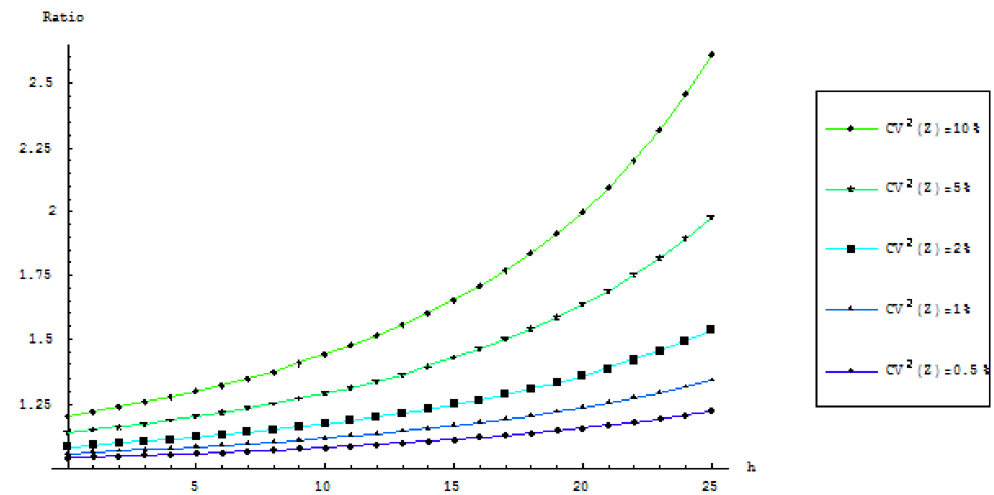


Figure 2: *Ratios* $\frac{q_\alpha^{Frailty}(\widetilde{Y})}{q_\alpha^{Indep}(\widetilde{Y})}$ *as a function of* h *for different values of* CV^2 *:* $0.5\%, 1\%, 2\%, 10\%$ *and* 20%.

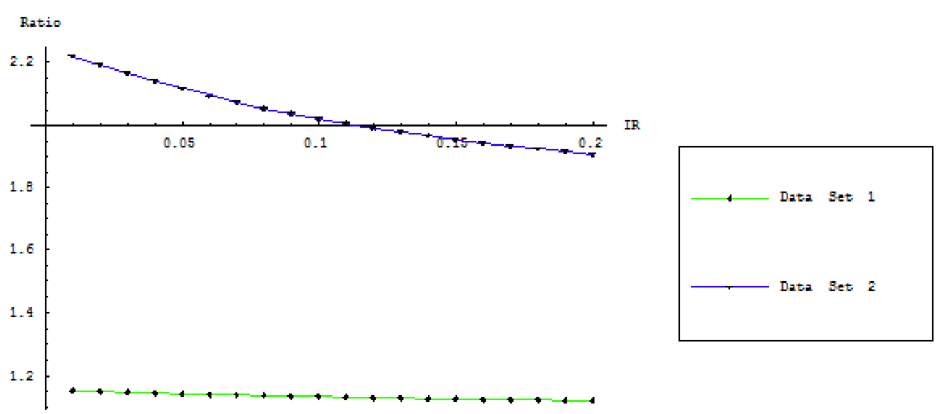


Figure 3: *Ratio* $\frac{q_\alpha^{Frailty}(\widetilde{Y})}{q_\alpha^{Indep}(\widetilde{Y})}$ *as a function of* i*, for the two Data Sets, representing the level of dependence.*

Gamma-Gompertz shared frailty model. A high interest rate environment puts less value on longer term liabilities, such as premia paid in t years, t being very large. On the other side, introduction of a dependence in the representation of the mortality implies that the individuals will live longer in average.

4.2.5 Impact of the dependence on the annuity portfolio structure

We now analyse the impact of the dependence on the distribution of capital between the cedant and the reinsurer. We have represented (1) the percentile $q_\alpha(Y)$ of the random aggregate retained loss of the insurer under the XL agreement, (2) the percentile $q_\alpha\left(\widetilde{Y}\right)$ of the aggregate liability of the reinsurer and (3) the percentile $\Pi_0 = q_\alpha\left(\widetilde{Y}+Y\right)$ of the aggregate liability of the insurer on the annuity portfolio if no XL arrangement was made. For the sake of simplicity, we make the assumption that the total capital needed for the annuity portfolio is measured using the percentile principle with a confidence level α.

Data set 1 (low dependence) : capital allocation for longevity risk

Age	**Model Type**	$q_\alpha(Y)$	$q_\alpha\left(\widetilde{Y}\right)$	Π_0	$q_\alpha\left(\widetilde{Y}\right)/\Pi_0$
$x=50$	Gompertz	$126,730$	$31,577$	$158,307$	19.94%
$x=60$	Gompertz	$111,198$	$13,082$	$124,280$	10.53%
$x=70$	Gompertz	$86,487$	$2,741$	$89,228$	3.07%
Age	**Model Type**	$q_\alpha(Y)$	$q_\alpha\left(\widetilde{Y}\right)$	Π_0	$q_\alpha\left(\widetilde{Y}\right)/\Pi_0$
$x=50$	Frailty	$128,273$	$34,322$	$162,595$	21.11%
$x=60$	Frailty	$113,803$	$14,984$	$128,786$	11.63%
$x=70$	Frailty	$89,982$	$3,457$	$93,439$	3.70%

Data set 2 (high dependence) : capital allocation for longevity risk

Age	**Model Type**	$q_\alpha(Y)$	$q_\alpha\left(\widetilde{Y}\right)$	Π_0	$q_\alpha\left(\widetilde{Y}\right)/\Pi_0$
$x=50$	Gompertz	$119,778$	$22,157$	$141,936$	15.61%
$x=60$	Gompertz	$100,421$	$7,392$	$107,813$	6.86%
$x=70$	Gompertz	$73,546$	$1,047$	$74,593$	1.40%

Age	Model Type	$q_\alpha(Y)$	$q_\alpha\left(\widetilde{Y}\right)$	Π_0	$q_\alpha\left(\widetilde{Y}\right)/\Pi_0$
$x = 50$	Frailty	$128,063$	$35,023$	$163,086$	21.48%
$x = 60$	Frailty	$114,101$	$15,905$	$130,007$	12.23%
$x = 70$	Frailty	$91,373$	$4,059$	$95,433$	4.25%

For the low dependence case, the total capital needed Π_0 for the annuity portfolio increased by an average +3.68% due to the introduction of the frailty. In parallel, the price of the XL reinsurance increased by an average +19.33%. As a result, the share of capital allocated to XL reinsurance, represented by $q_\alpha\left(\widetilde{Y}\right)/\Pi_0$, increased from +11.18% to +12.15%, due to the introduction of a frailty. For the high dependence case, the total capital needed for the annuity portfolio increased by an average +21.14% due to the introduction of the frailty. In parallel, the price of the XL reinsurance increased by an average +253.63%. As a result, the share of capital allocated to XL reinsurance, represented by $q_\alpha\left(\widetilde{Y}\right)/\Pi_0$, increased from +7.96% to +12.65%, due to the introduction of a frailty. Therefore, the introduction of a frailty increases the percentile of the total portfolio liability, and increases of the allocation of capital to XL reinsurance.

To assess the impact of dependence on the allocation of the capital between the cedant and the reinsurer, we have represented in Figure 4 the ratio :

$$\frac{q_\alpha^{Frailty}\left(Y+\widetilde{Y}\right)}{q_\alpha^{Indep}\left(Y+\widetilde{Y}\right)}$$

and we have represented in Figure 5 the % increase of share of capital allocated to XL reinsurance due to dependence :

$$\frac{q_\alpha^{Frailty}\left(\widetilde{Y}\right)/\Pi_0^{Frailty} - q_\alpha^{Indep}\left(\widetilde{Y}\right)/\Pi_0^{Indep}}{q_\alpha^{Indep}\left(\widetilde{Y}\right)/\Pi_0^{Indep}}$$

The total capital needed for the annuity portfolio is an increasing function of age x and of the level of dependence. The share of capital allocated to reinsurance is clearly increased for a higher dependence, but this % increase reaches a maximum between ages $x = 50$ and $x = 60$. For higer ages, the part of the XL reinsurance is still not significant enough in comparison with

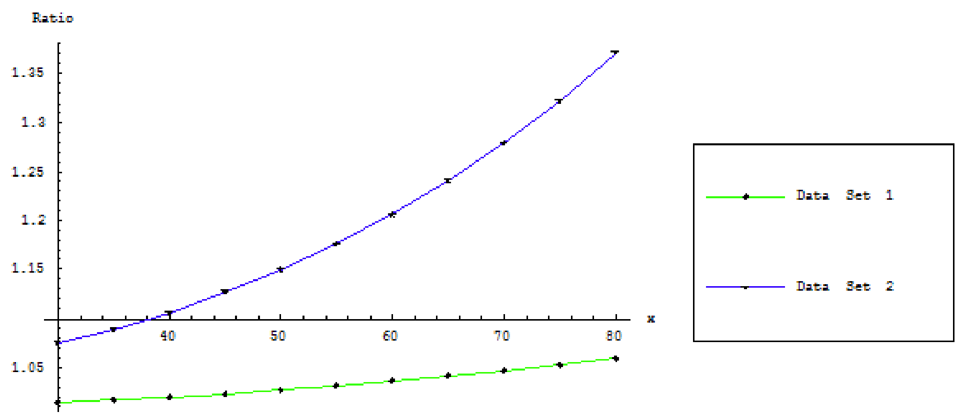


Figure 4: *Ratio* $\frac{q_\alpha^{Frailty}\left(Y+\widetilde{Y}\right)}{q_\alpha^{Indep}\left(Y+\widetilde{Y}\right)}$ *as a function of contract initiation age* x*, for different levels of dependence, corresponding to Data Set 1 (green curve) and Data Set 2 (blue curve).*

the total capital needed. The sharp decrease after $x = 60$ could indicate that the Gamma-Gompertz shared frailty model does not increase sufficiently the XL premium, to take into account the longevity risk.

5 Conclusion

This article is an attempt to represent mortality fluctuations and apply it to the pricing and risk management of life insurance portfolios. Modelling mortality risk is one of the challenges faced by life insurance companies. There is an impact of this risk on life insurance portfolios, a recently observed trend of mortality evolution and the forthcoming implementation of the new regulatory framework Solvency II.

There are two categories of mortality fluctuations models. The first are the intensity models, the second category concerns the lifetime dependence models. The model we propose, the shared frailty model, belongs to the second category. More precisely it is a model of common risk type of dependence. This lifetime dependence is generated by the common action on a population of some external and unobservable risk factors. A good meas-

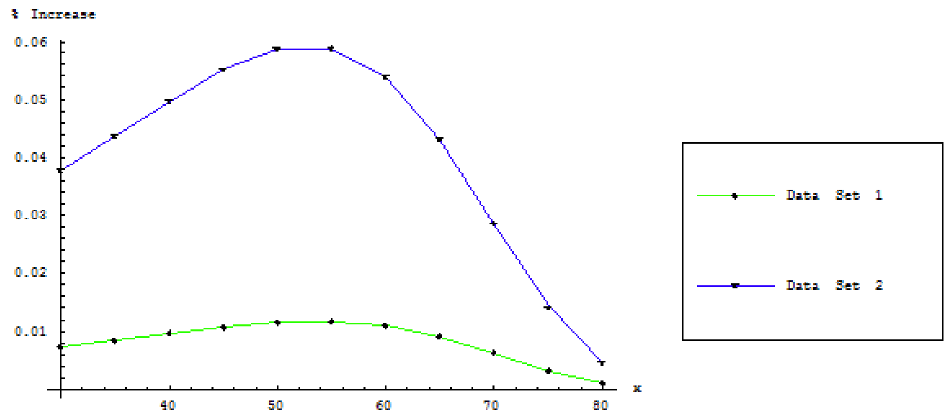


Figure 5: *Quantity* $\frac{q_\alpha^{Frailty}(\widetilde{Y})/\Pi_0^{Frailty} - q_\alpha^{Indep}(\widetilde{Y})/\Pi_0^{Indep}}{q_\alpha^{Indep}(\widetilde{Y})/\Pi_0^{Indep}}$ *as a function of contract initiation age* x*, for different levels of dependence, corresponding to Data Set 1 (green curve) and Data Set 2 (blue curve).*

ure of the dependence generated by the model is the coefficient of variability $CV^2(Z)$ of the frailty Z.

For the estimation, we have considered two population data sets, one with low dependence level, the other one with a high dependence level. We have calibrated the Gompertz(p, γ) model and the shared frailty model Gamma-Gompertz(p, γ, δ) on each of these two series. This approach is more refined than only moving the dependence parameter. On top of changing the dependence parameter δ, it allows to take into account the effect on the marginal risk parameters p and γ. A straight sensitivity analysis to δ would have ignored these compensation effects.

Then, we have applied the models to the pricing of an Excess of Loss (XL) reinsurance arrangement in the case of a portfolio of life annuities. In such contracts, the reinsurer pays the proceeds of the annuities exceeding a given term. Although we focus on annuity portfolios, our approach can easily be applied to life insurance portfolios or to other type of reinsurance contracts such as the Stop-Loss.

We have observed a significant impact of the dependence even in the case of a low dependence level. A dependence of 0.5%, as measured by the coefficient of variation of the frailty $CV^2(Z)$, leads to an average +20% increase in XL premia. A high dependence level of $CV^2(Z)$ equal to 10% leads to a +253% increase in the XL premia. These percentages of increase depend of the initiation age of the annuitants and of the term of the XL arrangement. A higher age or term leads to a higher increase in XL premia. In such case, the risk is more located in the tail of the aggregate risk distribution, and the introduction of a dependence increases the tail of the aggregate risk distribution.

The introduction of a frailty also leads to a change in the overall portfolio distribution between the cedant and the reinsurer. A larger proportion of the total capital required is transferred to the reinsurer when the dependence is considered. This increase of proportion is even higher when the dependence level is greater. In parallel, the total capital required, measured with a non-linear pricing method such as the percentile, was also increased. The figures were approximately +3.5% for a low dependence level of 0.5% and by approximately +20% for a large dependence level of 10%. These are significant values, since they concern the increase in value of the overall life annuity portfolio.

As a consequence, the frailty approach could be used to determine not only aggregate life insurance risk distributions, life reinsurance prices, but also all the aggregate risk measures used to compute the economic capital or minimum required capital. Those risk measures concern essentially the tail of the aggregate risk distribution. The use of a frailty model should also have some impact on asset and liability management, since there is a time frame of the dependence, as explained by Hougaard [2000].

We restricted ourselves to the simplest case of a shared unidimensional frailty. But the representation of a multivariate dependence by group of individual would give a better repartition of the dependence across risk classes. The representation of a non stationnary dependence would result in a better fit of extreme events and a more comprehensive time representation of the dependence, which could be interesting from an Asset and Liability management point of view. The analysis of other distributions for the frailty could lead to more flexibility in the representation of the aggregate risk distribution. We could imagine extension to more hybrid approaches, including a representation of the trend in longevity increase. These suggestions represent a territory for further research.

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