Contribution values for allocation of risk capital and for premium calculation

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Abstract

Starting from the capital allocation principle ('Euler principle') with Expected Shortfall as the underlying coherent risk measure, we introduce a new class of functionals in two random variables called contribution values, whose underlying risk measures are representable as Choquet integral w.r.t. distorted probabilities. A contribution value represents, roughly speaking, the aggregated expected values of a random variable X conditioned on the different quantiles of Y, the latter being weighted by a probability measure on the probability axis. We present a representation theorem for contribution values and prove inequalities which are crucial for capital allocation and, dually, for premium calculation within a portfolio of dependent (re-)insurance contracts.

Keywords Allocation of risk capital, insurance premium, coherent risk measure, Expected Shortfall, uniformisation, copula

1 Introduction

The capital allocation problem for a given (coherent) risk value ρ (i.e. the negative of a coherent risk measure) is the question of how the risk capital $\rho(Y)$ for a total risk $Y = \sum_{i=1}^{n} X_n$, $Y, X_1, \ldots, X_n \in L_1(P)$ should be assigned to the contributing risks $X_1, \ldots, X_n$ in a fair way, i.e. how to distribute the savings $\rho(Y) - \sum_{i=1}^{n} \rho(X_i)$ obtained by the diversification effect

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to the single risks $X_1, \dots, X_n$. A natural approach to represent a capital allocation is a functional $\Pi : L_1(P) \times L_1(P) \to \mathbb{R}$ satisfying

$$\Pi(X_i, Y) \geq \rho(X_i), \tag{1}$$
$$\Pi(X_1, Y) + \cdots + \Pi(X_n, Y) = \rho(Y). \tag{2}$$

The existence of a solution of the capital allocation problem given a coherent risk value directly follows from the well-known fact that a coherent risk value is the lower envelope of linear functionals (cf. e.g. [4]). Delbaen has shown in [4] that such a solution is generally not unique and may depend on the total risk Y. For the special case of the coherent risk value Expected Shortfall $\rho(Y) = \mathrm{ES}_\alpha(Y) = \int_0^\alpha \check{F}_Y(p)\, dp$ for level α, $0 < \alpha < 1$ with $\check{F}_Y$ being the pseudo inverse function of F_Y (cf. [17, p. 7]), Schmock suggested in [14] $\Pi(X, Y) := E(X | Y \leq q_\alpha(Y))$ as a natural solution to the capital allocation problem with $q_\alpha(Y)$ being the α-quantile of Y. This definition solves the capital allocation problem in the sense of satisfying conditions (1) and (2) only if $P(Y \leq q_\alpha(Y)) = \alpha$, i.e. if the distribution function of Y attains the value α. The proof of Schmock relies on Euler's Theorem for directional derivatives of homogenous functions, whence his capital allocation principle is sometimes called 'Euler principle'.

Using the functional $E(X | F_Y \circ Y = \cdot\,) : [0, 1] \to \mathbb{R}$ being defined by $E(X | F_Y \circ Y) = E(X | F_Y \circ Y = \cdot\,) \circ (F_Y \circ Y)$ (cf. [9, Theorem 4.2.8 and p. 267]) and by requiring to be $\sigma\left(\check{F}_Y\right)$-measurable, we show in Example 3.3 that Expected Shortfall ES_α has the representation

$$\mathrm{ES}_\alpha(Y) = \int_0^1 E(Y | F_Y \circ Y = \cdot\,) \cdot \frac{1}{\alpha} \cdot 1_{[0,\alpha]}\, d\lambda \tag{3}$$

and that the capital allocation principle $\Pi(X, Y) = E(X | Y \leq q_\alpha(Y))$ attributed to ES_α has the representation

$$E(X | Y \leq q_\alpha(Y)) = \int_0^1 E(X | F_Y \circ Y = \cdot\,) \cdot \frac{1}{\alpha} \cdot 1_{[0,\alpha]}\, d\lambda. \tag{4}$$

We replace the λ-density $\alpha^{-1} 1_{[0,\alpha]}$ by an arbitrary bounded density $\frac{dW}{d\lambda}$ of a probability measure $W \ll \lambda$, called a probability weight in the sequel. Thus we generalize the right hand side of Equation (4) in defining the contribution value $\Pi_W : L_1(P) \times L_1(P) \to \mathbb{R}$ by

$$\Pi_W(X, Y) := \int_0^1 E(X | F_Y \circ Y = \cdot\,)\, dW. \tag{5}$$

$\Pi_W(X, Y)$ then reflects the expected contributions of X to the different quantiles of Y, the latter weighted by W.

The coherent risk measures ρ, which can be represented as a Choquet integral w.r.t. a distorted probability, i.e. $\rho(Y) = \int Y\, d(\gamma \circ P)$ with γ :

$[0,1] \to [0,1]$ increasing, $\gamma(0) = 0$ and $\gamma(1) = 1$, are also representable by

$$\rho(Y) = \pi_W(Y) := \Pi_W(Y,Y)\,, \quad Y \in L_1(P)\,,$$

for some probability weight W with decreasing $\frac{dW}{d\lambda}$, related to the distortion function γ. Since we obtain in Theorem 4.2, that for a probability weight W with decreasing density $\frac{dW}{d\lambda}$ the capital allocation problem stated in Equations (1) and (2) is solved by Π_W, we finally obtain that Π_W is the natural solution of the capital allocation problem when the underlying coherent risk measure is representable by a Choquet integral w.r.t. a distorted probability measure.

The definition (5) of a contribution value is much more general than just representing an instrument for solving the capital allocation problem. When switching to probability weights with increasing density $\frac{dW}{d\lambda}$, the contribution value then solves the dual problem of (1) and (2) which is likewise important for apportioning the total premium for an insurance or reinsurance portfolio to it's components. The underlying risk value, i.e. the Choquet integral had already been applied to premium calculation in [5], [18], [19], [7]. These applications rely on the fact that the risk value of $X - EX$ is a volatility parameter, for example average absolute deviation and the Gini index belong to this class. Since there might be many more applications we first investigate the class of contribution values (5) without any monotonicity assumption for $\frac{dW}{d\lambda}$.

The paper is organized as follows. After fixing some notations and providing some preliminary results in the next section, our contribution functionals are defined in Section 3, the prominent examples are given and the essential properties including representation theorems are derived. Section 4 contains the main result of the paper, the proof of inequality (1) and its dual under the assumptions mentioned above. A practically important representation of the contribution functional by means of the copula of the random variables is aimed at in Section 5.

2 Preliminaries from measure and integration

Throughout the paper, we work on a basic probability space $(\Omega, \mathcal{A}, P)$. As usual $E(X|Y)$ denotes conditional expectation (w.r.t. P) of a random variable X given the random variable Y.

As increasing distribution function of a random variable X we employ the mean of the left and right continuous ones[1], i.e.

$$F_X(x) := \frac{P(X \leq x) + P(X < x)}{2}\,.$$

The inverse function of F_X is defined up to an at most countable subset of $[0,1]$ which will not lead to any problems throughtout its subsequent usage. In order to use functions being not only defined almost everywhere, we will use the pseudo inverse function $\check{F}_X$ of F_X, which, at any point $p \in [0,1]$, is the mean of the left and right continuous pseudo inverses at p (at the boundary points $p = 0$ and $p = 1$ of $[0,1]$ take the existing limit value). According to this definition, if the q-quantile of X is not unique, i.e. an interval, then $\check{F}_X(q)$ is the barycenter of this interval.

Like in [8] the random variable $U_X := F_X \circ X$ will be called the **uniformisation** of X. If F_X is continuous, then P^{U_X} is the uniform distribution on $[0,1]$. If F_X has a discontinuity at $x \in \mathbb{R}$ then the midpoint $F_X(x)$ of the interval $[P(X < x), P(X \leq x)]$ carries all weight $P^{U_X}(F_X(x)) = P(X \leq x) - P(X < x)$ of this interval. The uniformistion operator is idempotent, i.e. $U_{U_X} = U_X$.

We denote with $\mathcal{U}_X$ the sub-σ-algebra of the Borel σ-algebra $\mathcal{B}([0,1])$ of $[0,1]$ generated by $\check{F}_X$ and call it the **uniformisation σ-algebra** of X. This is the σ-algebra generated by the intervals $[P(X < x_1), P(X \leq x_2)]$, $x_1 \leq x_2$. A function $f : [0,1] \longrightarrow \mathbb{R}$ is $\mathcal{U}_X$-measurable iff it is $\mathcal{B}([0,1])$-measurable and constant on the intervals $]P(X < x), P(X \leq x)[$, $x \in \mathbb{R}$, of positive length. Clearly, $\mathcal{U}_X = \mathcal{B}([0,1])$ if F_X is continuous.

Proposition 2.1 *For $X, Y \in L_1(P)$ there exists a unique $\mathcal{U}_Y$-measurable function $E(X|U_Y = \cdot\,) : [0,1] \to \mathbb{R}$ in $L_1(\lambda|_{\mathcal{U}_Y})$ satisfying*

$$E(X|U_Y) = E(X|U_Y = \cdot\,) \circ U_Y.$$

Especially $P^{U_Y}|_{\mathcal{U}_Y} = \lambda|_{\mathcal{U}_Y}$ and

$$\int_0^1 f(p)\, dP^{U_Y}(p) = \int_0^1 f(p)\, dp \qquad \textit{for } f \in L_1(\lambda|_{\mathcal{U}_Y})\,.$$

For short we refer to $E(X|U_Y = \cdot\,)$ as conditional expectation as well, while correctly it should be denoted factorized conditional expectation.

Proof The existence of a $\mathcal{B}([0,1])$-measurable function $f_{X,Y} : [0,1] \to \mathbb{R}$ satisfying $E(X|U_Y) = f_{X,Y} \circ U_Y$ follows from Theorem 4.2.8 and the remarks on page 267 in [9]. By defining $E(X|U_Y = \cdot\,) := f_{X,Y} \circ E(\mathrm{id}_{[0,1]}|\check{F}_Y)$, we obtain the desired result. □

[1]This not very common definition has some advantage for the uniformisation σ-algebra below, one has not to care about the boundary points of the intervals defined by jumps of the distribution function. Furthermore this definition is the natural one if copulas of random vectors with non-continuous marginal distributions are involved (see [8]).

Example 2.1 *Let P be Lebesgue measure on $\Omega = [0,1]$ and $X = 1_{[.3,.7]}$, $Y = 1_{[0,.6]}$. Then $P^Y = .4\ \delta_0 + .6\ \delta_1$, where δ_y denotes Dirac measure at $y \in \mathbb{R}$. Similarly, $P^{U_Y} = .4\ \delta_{.2} + .6\ \delta_{.7}$. One easily computes $E(X|U_Y) = .5\ 1_{[0,.6]} + .25\ 1_{[.6,1]}$ and $E(X|U_Y = \cdot) = .5\ 1_{\{.7\}} + .25\ 1_{\{.2\}}$. The σ-algebra $\mathcal{U}_X \subset \mathcal{B}([0,1])$ is generated by $\{[0,.4],[.4,1]\}$. Finally $P^{U_Y}([0,.4]) = P^{U_Y}(.2) = .4$, $P^{U_Y}([.4,1]) = P^{U_Y}(.7) = .6$ and P^{U_Y}-a.e. $E(X|U_Y = \cdot) = .25\ 1_{[0,.4]} + .5\ 1_{[.4,1]}$. This version of $E(X|U_Y = \cdot)$ is $\mathcal{U}_Y$-measurable.*

Example 2.2 *If Y and F_Y are invertible functions and X is $\sigma(Y)$-measurable then $E\,(X|U_Y = p) = X \circ U_Y^{-1}(p)$.*

Lemma 2.2 *Let $X \in L_1(P)$ then $E(X|U_X = p) = \check{F}_X(p)$ for P^{U_X}-almost all $p \in [0,1]$.*

Proof By the definition of conditional expectation we know that $E(X|U_X)$ is the $\sigma(U_X)$-measurable function Z on Ω such that

$$\int_A Z\,dP = \int_A X\,dP \quad \text{for all } A \in \sigma(U_X)\,.$$

So, it is sufficient to show that these conditions hold for $Z = \check{F}_X \circ U_X$. This is plain since $\check{F}_X \circ U_X = \check{F}_X \circ F_X \circ X$ and P^X-a.e. we have $\check{F}_X \circ F_X = \mathrm{id}_{\mathbb{R}}$. □

From Choquet integration theory (see e.g. [6]) we use the notion of distorted probability [20]. A distortion function γ is a distribution function on the unit interval with $\gamma(0) = 0$, $\gamma(1) = 1$. Since in non-additive integration one has to employ decreasing distribution functions, we often will have to switch by conjugation. The distribution function

$$\overline{\gamma}(p) := 1 - \gamma(1-p)\,, \quad p \in [0,1]\,,$$

is called the **conjugate** of γ. Clearly, $\overline{\overline{\gamma}} = \gamma$ and γ is concave iff $\overline{\gamma}$ is convex. We get

$$\int_\Omega X\,d(\gamma \circ P) = \int_{-\infty}^{\infty} x\,d(\overline{\gamma} \circ F_X)(x)\,, \tag{6}$$

where the left hand integral is the Choquet integral w.r.t. the distorted probability $\gamma \circ P$, the right hand one the usual (Lebesgue-Stieltjes) integral on the real line w.r.t. the distribution function $\overline{\gamma} \circ F_X$.

$\mathcal{K}_X$ denotes the family of lower level sets $\{X < x\}$, $\{X \le x\}$ for $x \in \mathbb{R} \cup \{\infty, -\infty\}$. It is a chain w.r.t. set inclusion. Two random variables X, Y are comonotonic if $\mathcal{K}_X \cup \mathcal{K}_Y$ is a chain, too. For other characterisations see [6]. X, Y are called **strongly comonotonic** if $\mathcal{K}_X = \mathcal{K}_Y$. Strong comonotonicity is an equivalence relation, whereas simple comonotonicity is not. The equivalence classes of strong comonotonicity are convex cones.

Proposition 2.3 *Let $X, Y : \Omega \to \mathbb{R}$ be arbitrary functions.*

(i) There exists a function $\varphi : Y(\Omega) \to \mathbb{R}$ satisfying $X = \varphi \circ Y$ if $\sigma(X) \subset \sigma(Y)$.

(ii) $\mathcal{K}_X \subset \mathcal{K}_Y$ if and only if there exists an increasing function $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying $X = \varphi \circ Y$.

(iii) X and Y are strongly comonotonic if and only if there exists a strongly increasing function $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying $X = \varphi \circ Y$.

Proof (i) $Y^{-1}(y)$ with $y \in Y(\Omega)$ is an atom of $\sigma(Y)$, hence $\sigma(X) \subset \sigma(Y)$ implies $|X(Y^{-1}(y))| = 1$. Then the function $\varphi : Y(\Omega) \to \mathbb{R}$, $\varphi := X \circ Y^{-1}$ is well defined and solves the problem.

(ii) The if part is plain. Now suppose $\mathcal{K}_X \subset \mathcal{K}_Y$. Then $\sigma(X) = \sigma(Y)$ and, by part (i), $X = \varphi \circ Y$ for some $\varphi : Y(\Omega) \to \mathbb{R}$. Assume, contrary to φ being increasing, that $Y(\omega_1) < Y(\omega_2)$ and $X(\omega_1) > X(\omega_2)$. Then $\omega_2 \notin \{X \geq X(\omega_1)\} = \{Y \geq y_1\}$ for some y_1. But ω_2 should be contained in this set, since ω_1 is contained and $Y(\omega_2) > Y(\omega_1) \geq y_1$, a contradiction. Finally, φ can be extended from $Y(\Omega)$ to an increasing function on $\mathbb{R}$.

(iii) For showing the only if part we apply (ii) twice to get increasing functions $\varphi : Y(\Omega) \to X(\Omega)$ and $\psi : X(\Omega) \to Y(\Omega)$ which compose to the identities. So they are one-to-one, hence strongly increasing. □

3 Contribution value and risk value

A broad class of functionals Π is constructed as follows. Let W be a probability measure on the Borel σ-algebra $\mathcal{B}([0,1])$ of the unit interval and λ the uniform distribution, i.e. Lebesgue measure on $\mathcal{B}([0,1])$. We call W a **probability weight** if $W \ll \lambda$ with bounded λ-density. Then any Lebesgue integrable function is also W-integrable, $L_1(\lambda) \subset L_1(W)$. Throughout the article, F_W denotes the distribution function of the identity function on $[0,1]$ w.r.t. W.

Now we define the **contribution value for probability weight** W as

$$\Pi_W(X,Y) := \int_0^1 E(X|U_Y = p)\, dW(p)\,, \quad X, Y \in L_1(P).$$

$\Pi_W(X,Y)$ is real valued since $E(X|U_Y = \cdot\,) \in L_1(\lambda)$ by Lemma 2.1. Since conditional expectation $E(X|U_Y = \;\cdot\,)$ is determined only up to an additive nullfunction, we had to suppose $W \ll \lambda$ in order to get uniqueness for Π_W.

The corresponding **risk value**

$$\pi_W(X) := \Pi_W(X,X)\,, \quad X \in L_1(P),$$

for probability weight W is a Choquet integral.

Proposition 3.1 *(i) The application*

$$W \mapsto \gamma_W := \overline{F}_W$$

is a bijection between the set of probability weights and the set of Lipschitz-continuous distortion functions. Moreover, F_W is convex if and only if γ_W is concave and F_W is concave if and only if γ_W is convex.

(ii) Every risk value π_W has a representation as a Choquet integral w.r.t. a distorted probability with a Lipschitz-continuous distortion function and vice versa,

$$\pi_W(X) = \int_\Omega X \, d(\gamma_W \circ P)\,, \quad X \in L_1(P)\,. \tag{7}$$

Notice that expression (7) for π_W may still make sense if property $W \ll \lambda$ of a probability weight is dropped (see Example 3.2). Furthermore, the preceding proposition relates to every Choquet integral w.r.t. a Lipschitz-continuously distorted probability measure a probability weight and thus a contribution value. We will see in the next section that for those Choquet integrals being a coherent risk value, this attributed contribution value is the natural solution of the capital allocation problem (1) and (2).

Proof

(i) Lipschitz-continuity of the distribution function F_W (and thus of γ_W) is equivalent to W being a probability weight.

(ii) Since $W \ll \lambda$, F_W is continuous and so is γ_W. By Lemma 2.2, $\pi_W(X) = \int_0^1 E(X|U_X = p)\, dW(p) = \int_0^1 \check{F}_X \, dW$. The distribution function of $\check{F}_X$ w.r.t. the measure W equals $F_W \circ F_X$ a.e., whence the last integral equals $\int_{-\infty}^{\infty} x \, d(F_W \circ F_X)(x)$. Now apply (6). $\square$

Before providing some examples, the subsequent proposition states some elementary properties of π_W.

Proposition 3.2 *Given a probability weight W on the unit interval, the following properties hold for all $X, X_1, X_2 \in L_1(P)$:*

(i) $\pi_W : L_1(P) \to \mathbb{R}$ *is Lipschitz-continuous.*

(ii)
$$\pi_W(cX) = c\,\pi_W(X) \quad \text{for } c > 0.$$

(iii) If X_1, X_2 are comonotonic, then

$$\pi_W(X_1 + X_2) = \pi_W(X_1) + \pi_W(X_2)\,.$$

(iv) If the distribution function F_W of W is concave, then π_W is superadditive,

$$\pi_W(X_1 + X_2) \geq \pi_W(X_1) + \pi_W(X_2)\,.$$

Dually, π_W is subadditive if F_W is convex.

(v) If $F_{X_1} \geq F_{X_2}$, especially if $X_1 \leq X_2$, then $\pi_W(X_1) \leq \pi_W(X_2)$.

These properties imply that π_W is a coherent risk value in the sense of [2] if F_W is concave. The latter condition holds for Expected Shortfall but not for Value at Risk (cf. subsequent examples).

Proof

(i) The application $X \mapsto \check{F}_X \in L_1(\lambda)$ is Lipschitz-continuous by [6] Proposition 9.9 and $\check{F}_X \mapsto \pi_W(X) = \int_0^1 \check{F}_X\, dW$ is Lipschitz-continuous since W has bounded λ-density.

(ii) follows from Proposition 3.1 using positive homogeneity of the Choquet integral.

(iii) derives from the Choquet integral representation (7) of π_W and comonotonic additivity of the Choquet integral (e.g. [6]).

(iv) Since F_W is concave, the distortion γ_W in (7) is convex, whence the Choquet integral w.r.t. $\gamma_W \circ P$ is superadditive ([6]).

(v) is again implied by (7) and the definition of the Choquet integral. □

Example 3.1 *Let $W = \lambda$ be the uniform distribution on $[0,1]$, then Lemma 2.1 implies that $\Pi_W(X,Y) = \int E(X|U_Y = p)\, dP^{U_Y}(p) = EX$, the expected value of X.*

Example 3.2 *Let W be the Dirac measure δ_α at point $\alpha \in]0,1]$ (it is not absolutely continuous), then π_{δ_α} in (7) is* **Value at Risk** *for level α, $\pi_{\delta_\alpha}(X) = \mathrm{VaR}_\alpha(X)$. The corresponding distortion function in (7) is $\gamma_W = 1_{[1-\alpha,1]}$ (except for $\gamma_W(1-\alpha) = \frac{1}{2}$ according to our definition of the distribution function). The well known shortcoming of VaR_α is due to the fact that γ_W (and it's conjugate F_{δ_α}) is neither convex nor concave for $\alpha \in]0,1[$, see Proposition 3.3 (ix).*

Example 3.3 *Tasche stated in [17, p. 7] that the coherent risk measure* **Expected Shortfall** *for level α, $0 < \alpha < 1$, can easily be shown to have the representation $\mathrm{ES}_\alpha(Y) = -\frac{1}{\alpha}\int_0^\alpha \check{F}_Y(p)\, dp$. Actually, this implies $\mathrm{ES}_\alpha(Y) = \pi_W(Y)$ with $\frac{dW}{d\lambda} = \alpha^{-1} \cdot 1_{[0,\alpha]}$. In terms of Choquet integrals, we obtain $\mathrm{ES}_\alpha(Y) = \int Y\, d(\gamma_W \circ P)$ with $\gamma_W(p) = 0 \vee \frac{p-(1-\alpha)}{\alpha}$. Since γ_W is convex,* *one easily obtains the well-known fact that Expected Shortfall is coherent just by applying the elementary properties of Choquet integrals and Proposition 10.3 in [6]. It is easy to show that* $\Pi_W(X,Y) = E(X|Y \leq q_\alpha(Y))$ *with* $q_\alpha(Y)$ *being the* α*-quantile of* Y *in the case that the distribution function of* Y *attains* α*. Schmock proposed in [14] this functional to be a natural solution of the capital allocation problem* (1) *and* (2)*. For arbitrarily distributed* Y *the results in the next section will imply that* Π_W *is a natural solution of the capital allocation problem with Expected Shortfall as the underlying risk measure.*

Generalizing Example 3.3 we will show in Theorem 4.2 that w henever π_W is a coherent risk value that has a representation as a Choquet integral w.r.t. distorted probability measure with a Lipschitz-continuous distortion function, then Π_W is the natural solution of the capital allocation problem (1) and (2).

Example 3.4 *Let* W *be the measure with density* $1-\rho$ *on* $[0,1/2[$ *and* $1+\rho$ *on* $[1/2,1]$ *with a parameter* $0 \leq \rho \leq 1$*, then* π_W *is the* **absolute deviation premium principle** *of [5]. Similarly, the* **Gini premium principle** *[5] is generated with the quadratic distribution function* $F_W(p) = \rho p^2 + (1-\rho)p$ *on* $[0,1]$ *with parameter* $0 \leq \rho \leq 1$*. Both premium principles* π_W *are subadditive and share all good properties of a premium principle (see Proposition 3.3).*

Example 3.5 *Regard the absolute deviation principle in Example 3.4 for parameter* $\rho = 1$*, i.e.* $F_W(p) = 0 \vee (2p-1)$ *on* $[0,1]$*, then* $\pi_W(X - EX)$ *is average absolute deviation of* X *from median* $MX = \check{F}(\frac{1}{2})$*,*

$$\pi_W(X - EX) = \int_\Omega |X - MX|\, dP\,.$$

Similarly, for the Gini principle, take $\rho = 1$*, i.e.* $F_W(p) = p^2$ *on* $[0,1]$*, then* $\pi_W(X - EX)$ *is, up to normalization, the Gini index of* X*. See [5], [8].*

Hence, for centralized random variables, Π plays the role of a co-risk value associated with the risk value (volatility parameter) π. Or, formulated more abstractly, Π provides a substitute for the missing inner product in the Banach space $L_1(P)$ compared to the Hilbert space $L_2(P)$. As a substitute it does not share all properties of an inner product, for example it is not commutative and linear only in the first variable, as will be shown below.

Proposition 3.3 *Let* W *be a probability weight.*

(i) If X, Y *are independent then*

$$\Pi_W(X,Y) = EX.$$

(ii) If $Y_1, Y_2 \in L_1(P)$ are strongly comonotone then

$$\Pi_W(\,\cdot\,, Y_1) = \Pi_W(\,\cdot\,, Y_2).$$

Result (i) expresses the property, desired for co - risk values like covariance, that it is zero for independent centralized random variables. Result (ii) states that $\Pi_W(X, \,\cdot\,)$ is constant on the strong comonotonicity classes in $L_1(P)$ for any fixed $X \in L_1(P)$.

Proof (i) X and U_Y are independent, too, hence $E(X|U_Y) = EX$.
(ii) If Y_1 and Y_2 are strongly comonotonic then $U_{Y_1} = U_{Y_2}$ such that the result holds by definition of the contribution value. □

An important representation of contribution values is provided in the following proposition.

Proposition 3.4 *The functional $\Pi_W(\,\cdot\,, Y) : L_1(P) \to \mathbb{R}$ can be represented as an expected value w.r.t. the probability measure $Q_{W,Y}$ with $P|_{\sigma(Y)}$-density $h_{W,Y} := \frac{d(W|_{\mathcal{U}_Y})}{dP^{U_Y}} \circ U_Y$,*

$$\Pi_W(X, Y) = \int_\Omega X \, dQ_{W,Y}.$$

Proof The result holds by the following calculations using the transformation rule (third equality) and elementary properties of conditional expectation (last two equalities),

$$\begin{aligned}
\Pi_W(X, Y) &= \int_{[0,1]} E(X|U_Y = \cdot\,) \, dW|_{\mathcal{U}_Y} \\
&= \int_{[0,1]} E(X|U_Y = \cdot\,) \cdot \frac{d(W|_{\mathcal{U}_Y})}{dP^{U_Y}} \, dP^{U_Y} \\
&= \int_\Omega E(X|U_Y) \cdot \left(\frac{d(W|_{\mathcal{U}_Y})}{dP^{U_Y}} \circ U_Y \right) dP \\
&= \int_\Omega E\left(X \cdot \left(\frac{d(W|_{\mathcal{U}_Y})}{dP^{U_Y}} \circ U_Y \right) \middle| U_Y \right) dP \\
&= \int_\Omega X \cdot \left(\frac{d(W|_{\mathcal{U}_Y})}{dP^{U_Y}} \circ U_Y \right) dP. \qquad \square
\end{aligned}$$

We finish this section with a characterization of contribution values by some of their properties. These properties could also have been used to axiomatize contribution values.

Theorem 3.5 *Let W be a probability weight and π_W the corresponding risk value according to Proposition 3.1. A functional $\Lambda : L_1(P) \times L_1(P) \to \mathbb{R}$ satisfies for every $X, Y, Y_1, Y_2 \in L_1(P)$*

(i) $\Lambda(\,\cdot\,, Y) : L_1(P) \to \mathbb{R}$ *is monotone, continuous, and linear,*

(ii) $\Lambda(1_A, Y) = \pi_W(1_A)$ *for all* $A \in \mathcal{K}_Y$*,*

(iii) $\Lambda(X, Y) = \Lambda(E(X|Y), Y)$*,*

if and only if Λ *is the contribution value defined by* W*,*

$$\Lambda = \Pi_W\,.$$

Proof We first show that Π_W satisfies properties (i) to (iii). (i) is plain by the integral representation of $\Pi_W(\,\cdot\,, Y)$ given in Proposition 3.4. (iii) follows from $\sigma(Y)$-measurability of the the density function $h_{W,Y}$ in Proposition 3.4 so that

$$\begin{aligned}\Pi_W(E(X|Y), Y) = \int E(X|Y) \cdot h_{W,Y}\, dP &= \int E(X \cdot h_{W,Y}|Y)\, dP \\ &= \int X \cdot h_{W,Y}\, dP = \Pi_W(X, Y).\end{aligned}$$

We obtain (ii) by sequentially using properties Proposition 3.3 (ii), linearity of $\Pi_W(\,\cdot\,, Y)$, again Proposition 3.3 (ii), the definition of π_W, and comonotonic additivity of π_W (Proposition 3.2 (iii)),

$$\begin{aligned}\Pi_W(1_A, Y) &= \Pi_W(1_A, Y + 1_A) \\ &= \Pi_W(Y + 1_A, Y + 1_A) - \Pi_W(Y, Y + 1_A) \\ &= \Pi_W(Y + 1_A, Y + 1_A) - \Pi_W(Y, Y) \\ &= \pi_W(Y + 1_A) - \pi_W(Y) \\ &= \pi_W(1_A).\end{aligned}$$

Now suppose Λ satisfies properties (i) to (iii). From (i) and the Riesz Representation Theorem follows that for every $Y \in L_1(P)$ there exists a measure $Q_Y : \mathcal{A} \to \mathbb{R}_+$ representing $\Lambda(\,\cdot\,, Y)$ by $\Lambda(\,\cdot\,, Y) = \int \cdot\, dQ_Y$. From (ii) follows that Q_Y is a probability measure since $Q_Y(\Omega) = \Lambda(1, Y) = \pi_W(1) = 1$. Since, by (ii), Q_Y is uniquely defined on the π-system $\mathcal{K}_Y$, Q_Y and therefore also Λ is uniquely defined on $\sigma(Y)$ (cf. [3, Theorem 3.3]). By property (iii), Λ is then also uniquely defined on $L_1(P)$ which finishes the proof since Π_W has been shown above to satisfy properties (i) to (iii). $\Box$

Corollary 3.6 *Let* W *be a probability weight and* $X, Y \in L_1(P)$ *then*

$$\Pi_W(X, Y) = \pi_W(X) \quad \textit{whenever} \quad \mathcal{K}_X \subset \mathcal{K}_Y\,.$$

Proof The last paragraph in the proof of Theorem 3.5 shows, together with Proposition 3.1, that the decreasing distribution functions of X w.r.t. $\gamma_W \circ P$ and w.r.t. $Q_{W,Y}$ coincide, $\gamma_W \circ P(X \geq x) = \pi_W(1_{\{X \geq x\}}) = Q_{W,Y}(X \geq x)$,

$x \in \mathbb{R}$, and so do the integrals $\pi_W(X) = \int_\Omega X\, d(\gamma_W \circ P) = \int_\Omega X\, dQ_{W,Y} = \Pi_W(X,Y)$. $\square$

Let us discuss some features of properties (i) to (iii) in Theorem 3.5, which characterize contribution values.

Property (i) contains condition (2) of a capital allocation functional. The harder property (1) will be treated in the next section.

Property (ii), being under (i) and (iii) equivalent to the property stated in Corollary 3.6, states that the contribution $\Pi_W(X,Y)$ of X to Y equals the stand alone risk value $\pi_W(X)$ of X, if X is depending increasingly on Y in the sense that it is an increasing function of Y (Proposition 2.3). In other words, under increasing dependence of X on Y there is no incentive for the owner of X to join the portfolio Y.

Property (iii) states that $\Pi_W(\,\cdot\,,Y)$ is constant on the equivalence classes $\{X' \in L_1(P) \mid E(X'|Y) = E(X|Y)\}$, $X \in L_1(P)$, induced by conditional expectation given Y.

4 The superlinear and sublinear case

In this section, we will consider the special cases of probability weights W with concave resp. convex distribution function F_W or, equivalently, the distortion function γ_W being convex resp. concave. It will turn out, that the first case is sufficient to solve the capital allocation problem (1) and (2) whenever the underlying risk value is coherent and has a representation as a Choquet integral w.r.t. a distorted probability w.r.t. a Lipschitz-continuous distortion function. The latter case will turn out to be sufficient to solve the dual problem that is important in insurance mathematics.

Lemma 4.1 *Let $A \in \mathcal{A}$, $Z \in L_1(P)$, and $B \in \{\{Z < b\}, \{Z \le b\}\}$ for some $b \in \mathbb{R}$ such that $P(B) \ge P(A)$. Then*

$$\int_A Z\, dP \ge \int_B Z\, dP - (P(B) - P(A))b.$$

Proof Using $P(A \setminus B) + P(B) = P(A \cup B) = P(B \setminus A) + P(A)$, we get

$$\begin{aligned}
\int_A Z\, dP &= \int_{A\setminus B} Z\, dP + \int_{A\cap B} Z\, dP \\
&\ge \int_{A\setminus B} b\, dP + \int_{A\cap B} Z\, dP \\
&= \int_{B\setminus A} b\, dP - (P(B) - P(A))b + \int_{B\cap A} Z\, dP \\
&\ge \int_B Z\, dP - (P(B) - P(A))b. \qquad \square
\end{aligned}$$

Theorem 4.2 *Given a probability weight W with concave F_W then*

$$\Pi_W(X,Y) \geq \pi_W(X) \quad \textit{for all } X, Y \in L_1(P)\,.$$

Two special cases of the theorem can be proved easily. If X and Y are independent then the result follows from Proposition 3.3 (i) and Proposition 3.1. If X is $\sigma(Y)$ measurable, then from [6, Proposition 10.1] and the characterization of $Q_{W,Y}$ by its values on $\mathcal{K}_Y$ in Proposition 3.5 (iii) yield $Q_{W,Y} \geq \gamma_W \circ P$ and therefore the desired result.

Proof From concavity of F_W follows that $\frac{dW}{d\lambda}$ is decreasing. This density can be approximated by a sequence of functions which are convex combinations of densities of type $\alpha^{-1} 1_{[0,\alpha[}$. Then, by Lebesgue's Dominated Convergence Theorem, it is sufficient to show the hypotheses for these simple densities. Therefore, suppose $\frac{dW}{d\lambda} = \alpha^{-1} 1_{[0,\alpha[}$ with $0 < \alpha < 1$.

First, we regard the case that $\alpha = P(A)$ for some lower level set A of $E(X|U_Y)$. Then there exists an $a \in \mathbb{R}$ such that

$$\begin{aligned} & P(E(X|U_Y) < a) = \alpha < P(E(X|U_Y) < a + \varepsilon) && (8)\\ \text{or} \quad & P(E(X|U_Y) \leq a) = \alpha < P(E(X|U_Y) \leq a + \varepsilon) && (9) \end{aligned}$$

for all $\varepsilon > 0$. Moreover, let $b \in \mathbb{R}$ be the unique number satisfying

$$\begin{aligned} & P(X < b - \varepsilon) < \alpha = P(X < b) \\ \text{or} \quad & P(X \leq b - \varepsilon) < \alpha \leq P(X \leq b) && (10) \end{aligned}$$

for all $\varepsilon > 0$. Let $A \in \{\{E(X|U_Y) < a\}, \{E(X|U_Y) \leq a\}\}$ and $B \in \{\{X < b\}, \{X \leq b\}\}$. Applying Lemma 4.1 in the inequalities below first for

$Z = E(X|U_Y = \cdot\,)$ and then for $Z = X$, we obtain

$$\begin{aligned}
\alpha\Pi_{W_\alpha}(X,Y) &= \int_{[0,\alpha[} E(X|U_Y = \cdot\,)\,dP^{U_Y} \quad \text{see Lemma 2.1} \\
&\geq \int_{U_Y(A)} E(X|U_Y = \cdot\,)\,dP^{U_Y} \\
&\quad \text{since } U_Y(A) \text{ is a lower level set of } E(X|U_Y = \cdot\,) \\
&\quad \text{and } P^{U_Y}(U_Y(A)) = P^{U_Y}([0,\alpha[) \\
&= \int_A E(X|U_Y)\,dP \quad \text{since } E(X|U_Y) = E(X|U_Y = \cdot\,) \circ U_Y \\
&= \int_A X\,dP = \int_{A\setminus B} X\,dP + \int_{A\cap B} X\,dP \\
&= \int_{B\setminus A} X\,dP + \int_A X\,dP - \int_B X\,dP + \int_{A\cap B} X\,dP \\
&\geq \int_{B\setminus A} X\,dP - (P(B) - P(A))b + \int_{B\cap A} X\,dP \\
&= \int_{[0,P(B)[} \check{F}_X\,d\lambda - (P(B) - \alpha)b \quad \text{by (8) or (9)} \\
&= \int_{[0,\alpha[} \check{F}_X\,d\lambda \quad \text{since } \check{F}_X = b \text{ on }]\alpha, P(B)[\text{ by (10)} \\
&= \alpha\pi_{W_\alpha}(X).
\end{aligned}$$

Now suppose there does not exist an $a \in \mathbb{R}$ satisfying (8) or (9). Then there exists an $a \in \mathbb{R}$ satisfying

$$P(E(X|U_Y) < a) =: \alpha_- < \alpha < \alpha_+ := P(E(X|U_Y) \leq a)$$

since P is continuous from below for the left inequality and from above for the right one. We show that the affine function $f : [\alpha_-, \alpha_+] \to \mathbb{R}$,

$$f(\beta) := \int_{\{E(X|U_Y)<a\}} E(X|U_Y)\,dP + (\beta - \alpha_-)a,$$

dominates the function $\beta \mapsto \beta\pi_{W_\beta}(X)$ and is dominated by $\beta \mapsto \beta\Pi_{W_\beta}(X,Y)$ on $[\alpha_-, \alpha_+]$. This will finish the proof.

At the boundaries of the interval $[\alpha_-, \alpha_+]$ we obtain by the above sequence of inequalities

$$f(\alpha_-) = \int_{\{E(X|U_Y)<a\}} E(X|U_Y)\,dP \geq \alpha_-\pi_{W_{\alpha_-}}(X), \tag{11}$$

$$f(\alpha_+) = \int_{\{E(X|U_Y)\leq a\}} E(X|U_Y)\,dP \geq \alpha_+\pi_{W_{\alpha_+}}(X). \tag{12}$$

The function $\beta \mapsto \beta\pi_{W_\beta}(X)$ is convex as primitive of the increasing function $\check{F}_X$. Hence, by (11), (12), and f being affine, $f(\beta) \geq \beta\pi_{W_\beta}(X)$ for all $\beta \in [\alpha_-, \alpha_+]$.

Finally, for all $\beta \in [\alpha_-, \alpha_+]$, by Lemma 4.1,

$$\begin{aligned}\beta\Pi_{W_\beta}(X,Y) &= \int_{[0,\beta[} E(X|U_Y = \cdot\,)\,d\lambda \\ &\geq \int_{\{E(X|U_Y=\cdot\,)\leq a\}} E(X|U_Y = \cdot\,)\,dP^{U_Y} - (\alpha_+ - \beta)a = f(\beta). \qquad \square\end{aligned}$$

We now state the dual result of the preceding theorem which is needed when considering insurance premiums.

Corollary 4.3 *Given a probability weight W with convex F_W then*

$$\Pi_W(X,Y) \leq \pi_W(X) \quad \textit{for all } X, Y \in L_1(P)\,.$$

For the proof of this corollary, we will need the following technical result.

Lemma 4.4 *Let W^c be the probability weight with density $w^c(p) = w(1-p)$, $w := \frac{dW}{d\lambda}$. Then*

$$\Pi_W(X,-Y) = \Pi_{W^c}(X,Y)\,.$$

Proof The right continuous distribution function of $-Y$ is $P(-Y \leq y) = P(Y \geq -y) = 1 - P(Y < -y)$. A similar equation holds for the left continuous distribution function so that for the midpoint distribution functions $F_{-Y}(y) = 1 - F_Y(-y)$. Then we get $U_{-Y} = F_{-Y}(-Y) = 1 - F_Y(Y) = 1 - U_Y$ and finally $\Pi_W(X,-Y) = \int_0^1 E(X|U_{-Y} = p)w(p)\,dp = \int_0^1 E(X|1 - U_Y = p)w(p)\,dt = \int_0^1 E(X|U_Y = 1-p)w(p)\,dp = \int_0^1 E(X|U_Y = t)w(1-t)\,dt = \Pi_{W^c}(X,Y)$. $\square$

Proof (of Corollary 4.3) With the notations of Lemma 4.4, $F_{W^c} = \overline{F}_W$ is concave so that we derive from the theorem using Theorem 3.5 (i) and (ii)
$\Pi_W(X,Y) = \Pi_{W^c}(X,-Y) = -\Pi_{W^c}(-X,-Y)$
$\leq -\pi_{W^c}(-X) = -\Pi_{W^c}(-X,-X) = \Pi_{W^c}(X,-X) = \Pi_W(X,X) = \pi_W(X)$.
$\square$

We conclude this section with a discussion of the result stated in the theorem. For a coherent risk value $\pi = \int \cdot\, d(\gamma \circ P)$, i.e. for a risk value with convex distortion function γ, the existence of a solution of the capital allocation problem (1) and (2) is well-known (see also [11] for the relation to cooperative game theory where coherent risk values are shown to be essentially the same as exact cooperative games which are the lower envelope of their core). But generally, the solution is not unique. For instance, given the stand alone risk $Y = 1$, any probability measure Q in the core of $\gamma \circ P$, i.e. dominating $\gamma \circ P$, would solve the problem in the sense that the functional $\Lambda := \int \cdot \, dQ$ is a solution of the capital allocation problem. The proof of Theorem 3.5 implies that for any $Y \in L_1(P)$ uniqueness is given only on the subspace $L_1(P|_{\sigma(Y)})$ of $L_1(P)$. The contribution value presented here is then the distinct and natural solution that can be characterized to be the only one to satisfy property (iii) in Theorem 3.5.

5 Copula representation

Important, at least for applications, seems to be a copula representation of contribution values Π_W. First we show, that the representation in Proposition 3.4 can be given as an iterated integral.

For a random 2-vector (X, Y) on the probability space $(\Omega, \mathcal{A}, P)$ the (common) distribution function is $F_{X,Y}(x, y) := \frac{1}{2}(P(X \le x, Y \le y) - P(X < x, Y < y))$. We will call $C_{X,Y} := F_{U_X,U_Y}$ the **copula** of (X, Y). It determines uniquely the **copula distribution** $P^{(U_X,U_Y)}$ on the unit square. If X, Y are continuously distributed, then the present definition is the usual one (see e.g. [12]).

We need the conditional distribution (of id_Ω) given U_Y (see e.g. [9] 10.2),

$$P_{|U_Y}(A, p) := E\left(1_A | U_Y = p\right), \quad A \in \mathcal{A},\ p \in [0, 1]\,. \tag{13}$$

Proposition 5.1 *Suppose that Ω is a Polish space and $\mathcal{A}$ it's Borel σ-algebra. Let $Y \in L_1(P)$ and W a probability weight. Then, for P^{U_Y}-almost all $p \in [0, 1]$, $P_{|U_Y}(\,\cdot\,, p)$ is a probability measure, absolutely continuous w.r.t. P, and*

$$\Pi_W(X, Y) = \int_0^1 \int_\Omega X(\omega)\, g_Y(\omega, p)\, dP(\omega)\, dW(p)\,, \quad X \in L_1(P)\,. \tag{14}$$

where $g_Y(\,\cdot\,, p)$ denotes the P-density of $P_{|U_Y}(\,\cdot\,, p)$.

Hence, the measure $Q_{Y,W}$ is the product of W with the kernel $P_{|U_Y}$, i.e. with the family $\{P_{|U_Y}(\,\cdot\,, p) | p \in [0, 1]\}$ of probability measures.

Proof By [9] theorems 10.2.2 (which requires a Polish space) and 10.2.5, the conditional distribution $P_{|U_Y}(\,\cdot\,, p)$ is a probability measure for P^{U_Y}-almost all $p \in [0, 1]$ and

$$E(X|U_Y = p) = \int_\Omega X(\omega)\, P_{|U_Y}(d\omega, p) \quad \text{for} X \in L_1(P)\,.$$

Absolute continuity is plain from (13). □

In order to minimize technicalities with the copula we make, for the rest of this section, the following

Assumptions Ω is a Polish space, $\mathcal{A}$ it's Borel σ-algebra and all random variables $X, Y, \ldots$ on $(\Omega, \mathcal{A}, \mathcal{P})$ are continuously distributed.

Theorem 5.2 *For a probability weight W and $X, Y \in L_1(P)$,*

$$\Pi_W(X, Y) = \int_{-\infty}^{\infty} x\, d(\gamma_{W,X,Y} \circ F_X)(x)\,,$$

where $\gamma_{W,X,Y}(t) := \int_{\{U_X \le t\}} h_{W,Y}\, dP$.

For fixed $t \in [0,1]$, the partial derivative $\frac{\partial}{\partial p} C_{X,Y}(t,p)$ of the copula exists for almost all $p \in [0,1]$ and

$$\gamma_{W,X,Y}(t) = \int_0^1 \frac{\partial}{\partial p} C_{X,Y}(t,p)\, dW(p)\,. \tag{15}$$

One easily checks that $\gamma_{W,X,Y}(t) = t$ if X, Y are independent (cf. Proposition 3.3 (i)) and that $\gamma_{W,X,Y} = F_W$ if X, Y are comonotonic (less than strongly comonotonic as in Proposition 3.3 (ii) but under the assumption of X, Y being continuously distributed).

Proof $\{U_X \le F_X(x)\} = \{X \le x\}$, hence $\gamma_{W,X,Y} \circ F_X(x) = \int_{\{X \le x\}} h_{W,Y}\, dP$ $= Q_{W,Y}(X \le x)$ is the distribution function of X w.r.t. $Q_{W,Y}$ and (Proposition 3.4)

$$\int_{-\infty}^{\infty} x\, d(\gamma_{W,X,Y} \circ F_X)(x) = \int_\Omega X\, dQ_{W,Y} = \Pi_W(X, Y)\,.$$

For proving the representation of $\gamma_{W,X,Y}$ by means of the copula, we apply Proposition 3.4 and (14) to get $\gamma_{W,X,Y}(t) = \Pi_W(1_{\{U_X \le t\}}, Y) = \int_0^1 P_{|U_Y}(U_X \le t\,,\, p)\, dW(p)$. Finally, existence of the partial derivative of the copula as claimed is well known ([12] Theorem 2.2.7) and one easily computes

$$\frac{\partial}{\partial p} C_{X,Y}(t,p) = P_{|U_Y}(U_X \le t\,,\, p)$$

for those p for which the partial derivative exists. For this purpose recall $C_{X,Y}(t,p) = P(U_X \le t, U_Y \le p)$. □

Corollary 5.3 *(Hummel [10]) For the probability weight W_α of Expected Shortfall (Example 3.3) and $X, Y \in L_1(P)$,*

$$\Pi_{W_\alpha}(X, Y) = \int_{-\infty}^{\infty} x\, d(\gamma_{\alpha,X,Y} \circ F_X)(x)\,,$$

where $\gamma_{\alpha,X,Y}(t) := \frac{1}{\alpha} C_{X,Y}(t, \alpha)$, $t \in [0,1]$.

We get an alternative proof of Theorem 4.2 under the additional assumptions of this section.

Proof of Theorem 4.2 The reduction to the Expected Shortfall case is the same as in the proof in Section 5. So, we suppose that $W = W_\alpha$. Since X is comonotonic with itself, the copula $C_{X,X}$ is the upper Fréchet-Höffding copula, so that $C_{X,X} \geq C_{X,Y}$ [12]. In the Expected Shortfall case we get, with the notations of Corollary 5.3, $\gamma_{\alpha,X,X} \geq \gamma_{\alpha,X,Y}$ so that

$$\begin{aligned}\Pi_{W_\alpha}(X,Y) &= \int_{-\infty}^{\infty} x\, d(\gamma_{\alpha,X,Y} \circ F_X)(x) \\ &\geq \int_{-\infty}^{\infty} x\, d(\gamma_{\alpha,X,X} \circ F_X)(x) = \pi_{W_\alpha}(X).\end{aligned}$$

□

6 Conclusion and outlook

We have generalized Expected Shortfall and the associated contribution value. Some work has to be done for the copula representation if the random variables are not continuously distributed, this case being relevant for applications. But our definitions, especially that of the uniformisation σ-algebra, are prepared to work with this case as well (see Footnote 1). Challenging is the general form of Tasche's Theorem that $\Pi_W(\,\cdot\,,Y)$ is the Gateau derivative of π_W at Y. There are examples that this does not hold for all combinations of X and Y, but it seems to hold generically, since Tasche proved it if some conditional densities exist.

With respect to the applications we have constructed the canonical functional Π_W, associated with a coherent risk value representable as an expectation w.r.t. a distorted probability, as a model for capital allocation within the components of a risky portfolio. The dual case with a concave in place of a convex distortion is suited for the apportionment of (re-)insurance premiums to the components of an insurance portfolio. But our model allows general (non-convex or non-concave) distortions with, as we hope, further interesting applications. The axiomatization in Theorem 3.5 will help to check if our model is suited for envisioned applications.

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