COMPARISON BETWEEN SOME PENSION FUNDING SCHEMES IN A DETERMINISTIC CONTINUOUS TIME ENVIRONMENT

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Abstract

Amongst the various funding methods applied for a defined benefit pension plan, two important classes can be considered: the *unit credit cost* method and the *individual level premium* method.

The unit credit cost methodology is built on a fixed level of actuarial liability (from which the contributions due are derived); the individual level premium philosophy is directly expressed in terms of contributions stability.

The purpose of this paper is to compare from a theoretical perspective the two methods using closed forms based on classical calculus. We obtain explicit results for the contributions in a continuous dynamic economy and analyze the effects of growing salaries as well as the consequences of financial gains or losses. The main conclusion is the danger of individual level premium methods generally leading to non-bounded contributions in this continuous model and the advantage to adopt unit credit valuation as recommended by the IAS norms.

Key words: pension funding, unit credit cost, individual level premium, IAS norms

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1. Introduction

Pension funding methods are more than ever a key issue for actuaries especially in the context of the so-called "pay as you go" systems crisis. The demographic changes expected in our developed countries in the next decades represent a major challenge for the public social security systems, and funding methods seem to, at least partially, offer an adequate response to these challenges (Trowbridge (1952), Collinson (2001)).

At the same time financial markets have in recent years shown an extraordinary volatility, inducing significant solvency problems for many pension funds. In this context, although a lot of pension plans are nowadays in defined contributions, and this way transfer market risk to the affiliates, many pension plans still use defined benefits. Actuaries have to thoroughly analyse the various possible funding methods for these plans and compare the contributions evolution under different scenarios (Devolder (2005), Dufresne(1994)).

Two important families of funding methods can be considered: the *unit credit cost* approach and the *individual level premium* approach.

The unit credit cost methodology is based on a linear growth of the funded benefit (from which the contributions to pay are derived); it has become an increasingly standard method used, for instance, by international standard accounting norms (FAS and IFRS), although not necessarily imposed by local regulations (Anderson (1992), Berin (1986)).

The individual level premium philosophy originates from classic life insurance computations; the aim of the method is to induce constant contributions. If such stability is obtained in a static economy, it turns out that it disappears completely under dynamic conditions (Bowers et all (1986)).

The purpose of this paper is to analytically compare the contributions generated by these two methods in a continuous deterministic environment, from a theoretical perspective. Country-specific laws and regulations are not considered here.

Simple assumptions are used to focus on the main effect of the methods and to obtain closed forms of the contributions. In particular, classical calculus tools are used to identify a major property of the individual level premium principle (which is not present in unit credit): the contributions are non-bounded for continuous change of benefits closed to retirement. In practice, regular contributions are paid instead of continuous density; then the property of unbounded density can be translated in a huge increase of regular contributions. This way, the method initially meant to generate stable contributions leads to no stability at all closed to retirement.

The paper is structured as follows:

- section 2 compares the unit credit and individual level premium methods in a static environment (with constant benefits and no actuarial gain or loss) and in a continuous time framework;
- section 3 introduces a dynamic effect on the benefits;
- section 4 analyses the impact of differences between the actuarial discount rate and the real return on assets; and
- section 5 illustrates all these situations numerically.

2. *Unit credit and individual level premium in a static economy*

Let us consider a defined benefit pension plan. The objective is to look at the evolution of the contributions for a plan affiliate from affiliation age (t=0) to retirement age (t=T), using either unit credit cost methods or individual level premium methods in a continuous time model; in particular, we use contribution density instead of regular premiums. This will allow us to use the machinery of calculus in order to obtain easy formulations and detect fundamental behaviours.

In this section we consider a constant benefit (e.g. a fixed amount).
In the next sections, we will work with variable benefits, essentially linked to salaries.

In order to focus on the financial aspect, we will not take into account any mortality effect before retirement age.

We will use the following notations and assumptions:

- t = valuation time (with: $0 \leq t \leq T$);
- K= benefit to be paid or financed (in case of an annuity) at retirement age (t=T);
- $\pi(t) =$ contribution density at time t;
- $AL(t) =$ actuarial liability at time t;
- $F(t) =$ assets at time t;
- r = actuarial force of interest; in this section, it is assumed to be equal to the return on the assets.

2.1. Unit Credit cost method

Under the unit credit cost method, the main driver is the actuarial liability level. We consider here the unit credit cost method with service proration. The purpose of this method is to have at all times an amount of actuarial liability (prospective reserve) equal to:

$$AL_{UC}(t) = \frac{t}{T} K\ e^{-r(T-t)} \tag{1}$$

At the same time, the assets (retrospective reserve) can be expressed as the accumulation of the past contributions:

$$F(t) = \int_0^t \pi(s) e^{r(t-s)}\, ds$$

The equivalence between prospective and retrospective reserves gives the following integral equation for the contribution rate:

$$\frac{t}{T} K e^{-r(T-t)} = \int_0^t \pi(s) e^{r(t-s)} ds \tag{2}$$

Deriving this expression gives the explicit value of the contribution in unit credit cost:

$$\pi_{UC}(t) = \frac{1}{T} K e^{-r(T-t)} \tag{3}$$

The contributions follow an exponential growth.

2.2. Individual level premium method

The main driver under the individual level premium method is the contribution level. The purpose of the method is to define a constant contribution level. The equivalence principle can be written at maturity between:

- the actuarial liability to obtain:

$$AL_{LP}(T) = K$$

- the accumulated assets under a constant contribution rate:

$$F(T) = \int_0^T \pi e^{r(T-t)}\,dt$$

which gives the value of the constant contribution density:

$$\pi_{LP} = \frac{K r e^{-rT}}{(1-e^{-rT})} \qquad (4)$$

We can compute the t* moment, where the contributions in unit credit (equation (3)) and level premium (equation (4)) are equal:

$$t^* = \frac{1}{r} \ln\left(\frac{rT}{1-e^{-rT}}\right)$$

We can also compare the two actuarial liability levels.
Taking equation (4) into account, the actuarial liability in the individual level premium method becomes:

$$AL_{LP}(t) = \int_0^t \pi_{LP}\; e^{r(t-s)}\,ds = \frac{K r e^{-rT}}{(1-e^{-rT})} \int_0^t e^{r(t-s)}\,ds \; = K e^{-r(T-t)} \frac{1-e^{-rt}}{1-e^{-rT}} \qquad (5)$$

It is easy to see that: $AL_{UC}(t) \le AL_{LP}(t) \quad (0 \le t \le T)$

Indeed, given the equations (1) and (5), we have to prove that:

$$\frac{t}{(1-e^{-rt})} \le \frac{T}{(1-e^{-rT})} \qquad (\,0 \le t \le T)$$

or that the function $f(t) = \frac{t}{(1-e^{-rt})}$ is a non decreasing function.
The first derivative of this function is given by:

$f'(t) = e^{-rt} \frac{e^{rt} - (1+rt)}{(1-e^{-rt})^2}$ which is positive (taking into account the development of the exponential function).

So the comparison between the two methods for fixed benefits can be summarized as follows:

PROPOSITION 2.1.:

(i) *In terms of contributions*:

$$\pi_{UC}(t) \leq \pi_{LP}(t) \quad \text{for} \quad 0 \leq t \leq t^*$$
$$\pi_{UC}(t) \geq \pi_{LP}(t) \quad \text{for} \quad t \geq t^*$$

(ii) *In terms of actuarial liability*:

$$AL_{UC}(t) \leq AL_{LP}(t)$$

3. Unit credit and individual level premium in a dynamic economy

We assume here that the benefit K is no longer constant over time. As in the case of most defined benefit plans, the benefit is not constant. It increases when salaries change (e.g. final salary pension plan).

We will denote by K(t) the estimate of the benefit to be paid at time T based on the level of salary known at time t. The actuarial liability at time t becomes now:

$$AL_{UC}(t) = \frac{t}{T} K(t)\, e^{-r(T-t)}$$

3.1. Unit Credit Cost method

Using the methodology used in section 2.1., we get the following integral equation:

$$\frac{t}{T} K(t) e^{-r(T-t)} = \int_0^t \pi(s) e^{r(t-s)} ds$$

Taking the derivative we obtain the general solution:

$$\pi_{UC}(t) = \left(\frac{1}{T} K(t) + \frac{t}{T} \frac{\partial K}{\partial t}\right) e^{-r(T-t)}$$

We can consider the particular case of an exponential benefits growth generated by a constant salaries evolution rate for instance:

$$K(t) = K\,e^{gt}$$

The contributions become:

$$\pi_{UC}(t) = \frac{K}{T} e^{gt}\,(1 + g\,t)\,e^{-r(T-t)} \tag{6}$$

This contribution density is a non-decreasing function, but remains bounded. The contribution density at time T is given by:

$$\pi_{UC}(T) = \frac{K}{T} e^{gT}\,(1 + g\,T)$$

For a linear function of benefits:

$$K(t) = K_0 + K_1\,t$$

the contributions are given by:

$$\pi_{UC}(t) = \frac{1}{T}(K_0 + 2K_1 t)\; e^{-r(T-t)} \tag{7}$$

and the contribution density at time T becomes :

$$\pi_{UC}(T) = \frac{1}{T}(K_0 + 2K_1 T)$$

3.2. Projected Unit Credit Cost method:

The method is exactly the same as in the unit credit cost method, except that we now use an estimate of the final benefit taking into account salaries projection till retirement.

If the last salary estimate remains valid during the entire career, we can use the unit credit cost formula with constant benefits (section 2.1); if the estimate has to be adapted, we are in the case of variable benefits of section 3.1.

3.3. Individual level premium method

The differential equation of the contribution density can be obtained by the equivalence principle between:

- the accumulated assets at time t (retrospective reserve):

$$F(t) = \int_0^t \pi_{LP}(s)\, e^{r(t-s)}\, ds$$

- and the actuarial liability at time t (prospective reserve) based on a constant contribution rate from time t to maturity:

$$\begin{aligned} AL_{LP}(t) &= K(t)\, e^{-r(T-t)} - \pi_{LP}(t) \int_t^T e^{-r(s-t)}\, ds \\ &= K(t)\, e^{-r(T-t)} - \pi_{LP}(t) \frac{(1-e^{-r(T-t)})}{r} \end{aligned}$$

So, the contribution density is now the solution to the following integral equation:

$$\int_0^t \pi_{LP}(s)\, e^{-rs}\, ds + \pi_{LP}(t) \frac{e^{-rt} - e^{-rT}}{r} = K(t)\, e^{-rT}$$

Deriving this expression gives:

$$\pi_{LP}(t)\, e^{-rt} + \pi_{LP}(t)(-e^{-rt}) + \frac{\partial \pi_{LP}}{\partial t} \frac{e^{-rt} - e^{-rT}}{r} = e^{-rT} \frac{\partial K}{\partial t}$$

or:

$$\frac{\partial \pi}{\partial t} = r \frac{\frac{\partial K}{\partial t}}{(e^{r(T-t)} - 1)} \qquad (8)$$

The initial condition of this differential equation is (equation (4)):

$$\pi_{LP}(0) = \frac{K(0) r\, e^{-rT}}{(1-e^{-rT})}$$

The solution is then given by:

$$\pi_{LP}(t) = r\left(\frac{K(0)}{e^{rT}-1} + \int_0^t \frac{\partial K}{\partial s}\frac{1}{(e^{r(T-s)}-1)}ds\right)$$

We can see directly that we will have serious problems with this method in the neighbourhood of the maturity.

For instance, let us consider a linear benefit growth till time T-h, followed by a constant benefit till time T (no more salary increase at the end of the career):

$$\begin{aligned} K(t) &= K_0 + K_1 t & (0 \le t \le T-h) \\ &= K_0 + K_1(T-h) & (T-h \le t \le T) \end{aligned}$$

The contribution density becomes for $0 \le t \le T-h$:

$$\pi_{LP}(t) = \pi_{LP}(0) + K_1 r \int_0^t \frac{1}{(e^{r(T-s)}-1)}ds$$

Direct calculations give the final form of the contributions:

$$\pi_{LP}(t) = \pi_{LP}(0) + K_1\left(-r\,t + \ln\left(\frac{e^{rT}-1}{e^{r(T-t)}-1}\right)\right) \tag{9}$$

The contribution density at time T-h is:

$$\pi_{LP}(T-h) = \pi_{LP}(0) + K_1\left(-r(T-h) + \ln\left(\frac{e^{rT}-1}{e^{rh}-1}\right)\right)$$

which is not bounded for $h \to 0$.

The result we obtain is an explosion of this funding method when the benefits are continuously adapted till maturity. This phenomenon was not present in the unit credit cost method.

We can also consider an exponential benefit growth; let us consider for the sake of simplicity that the growth of the benefit is equal to the discount rate:

$$\begin{aligned} K(t) &= K_0\ e^{rt} \qquad\qquad (\ 0 \le t \le T-h) \\ &= K_0\ e^{r(T-h)} \qquad (T-h \le t \le T) \end{aligned}$$

The contribution density becomes for $0 \le t \le T-h$:

$$\pi_{LP}(t) = \pi_{LP}(0) + K_0 r^2 \int_0^t \frac{e^{rs}}{(e^{r(T-s)}-1)} ds$$

And after calculation:

$$\pi_{LP}(t) = \pi_{LP}(0) + K_0 r((1-e^{rt}) + e^{rT}\ln(\frac{e^{rT}-1}{e^{rT}-e^{rt}}))$$

Once again the rate is not bounded when approaching retirement.

In the general case where the benefit growth (g) and the discount rate (r) are different, we obtain the following integral to be computed numerically:

$$\pi_{LP}(t) = \pi_{LP}(0) + K_0 r \int_0^t \frac{g e^{gs}}{(e^{r(T-s)}-1)} ds \qquad (10)$$

We can summarize the results of this section in the following proposal:

PROPOSITION 3.1.:

When the benefits over the career are increasing and bounded functions with bounded first derivative:

(a) *the contributions in unit credit cost* are bounded and given at any time by:

$$\pi_{UC}(t) = (\frac{1}{T}K(t) + \frac{t}{T}\frac{\partial K}{\partial t})e^{-r(T-t)}$$

(b) *the contributions in individual level premium* are generally not bounded and given at any time by:

$$\pi_{LP}(t) = r(\frac{K(0)}{e^{rT}-1} + \int_0^t \frac{\partial K}{\partial s}\frac{1}{(e^{r(T-s)}-1)} ds)$$

4. *Financial surplus*

In the previous section the main assumption was the coincidence between the actuarial discount rate and the actual rate of return on the assets. We will now eliminate this assumption and make a clear distinction between:

-*the liability side:* r = actuarial force of interest used as discount rate

-*the asset side*: $\delta(t) =$ effective rate of return on assets at time t.

Two particular situations will be analysed more in detail in section 4.2:

Case 1 - constant difference:

$$\delta(t) = r + \Delta \quad \text{for } 0 \le t \le T$$
$$(\text{with } \Delta \ne 0)$$

(with asset and liability rates parallel but not equal).

Case 2 - jump:

$$\delta(t) = r \quad \text{for } 0 \le t < t_1$$
$$\delta(t) = r + \Delta \quad \text{for } t \ge t_1$$
$$(\text{with } \Delta \ne 0)$$

(At the beginning, asset and liability rates are equal but there is a shock on the asset return at some future time).

We will directly consider the general case with variable benefits. The main purpose is to look at the way the two methods react.

4.1. General form of the rates of return

We consider here a general form of the actual rates of return for the function δ.

4.1.1. Unit Credit cost method

The basic equivalence (equation (2)) becomes:

$$\frac{t}{T}K(t)\,e^{-r(T-t)} = \int_0^t \pi(s)\exp(\int_s^t \delta(u)\,du)\,ds$$

Taking the derivative gives:

$$\pi(t) = (\frac{1}{T}K(t) + \frac{t}{T}\frac{\partial K}{\partial t})\,e^{-r(T-t)} + r\,(\frac{t}{T}K(t)\,e^{-r(T-t)}) - \delta(t)\int_0^t \pi(s)\exp(\int_s^t \delta(u)\,du\,)\,ds$$

$$= (\frac{1}{T}K(t) + \frac{t}{T}\frac{\partial K}{\partial t})\,e^{-r(T-t)} + (r - \delta(t))(\frac{t}{T}K(t)\,e^{-r(T-t)})$$

The contribution has three parts :

1. the unit of the year
2. the effect of the variation of benefits on the past services
3. the effect of the difference between the discount rate and the rate of return.

4.1.2. Individual level premium method

The assets are now given by:

$$F(t) = \int_0^t \pi(s)\,(\exp\int_s^t \delta(u)\,du)\;ds$$

and the actuarial liability by:

$$AL(t) = K(t)\,e^{-r(T-t)} - \pi(t)\frac{1-e^{-r(T-t)}}{r}$$

The contribution rate is the solution of the integral equation:

$$\int_0^t \pi(s)\,(\exp\int_s^t \delta(u)\,du)\,ds + \pi(t)\frac{1-e^{-r(T-t)}}{r} = K(t)\,e^{-r(T-t)}$$

Taking as usual the derivative, after some manipulations we obtain the following differential equation of the contribution (equivalent to equation (8)):

$$\frac{\partial\pi}{\partial t} = \pi(t)\,(\delta(t)-r) + r\frac{(\frac{\partial K}{\partial t} + K(t)(r-\delta(t))}{(e^{r(T-t)}-1)}$$

with initial condition:

$$\pi_{LP}(0) = \frac{K(0)r\,e^{-rT}}{(1-e^{-rT})}$$

The solution is given by:

$$\pi(t) = \pi(0)\exp(\int_0^t(\delta(s)-r)\,ds) + r\int_0^t \frac{(\frac{\partial K}{\partial s} + K(s)(r-\delta(s)))}{(e^{r(T-s)}-1)}\exp(\int_s^t(\delta(u)-r)du)\ ds$$

4.2. Particular cases of evolution of the rates of return

Let us consider first the following case with a constant benefit K:

Case 1 - constant difference:

$$\delta(t) = r + \Delta \qquad \text{for } 0 \le t \le T$$
$$(\text{with } \Delta \neq 0)$$

The unit credit cost method gives the following contribution:

$$\pi(t) = (\frac{1-t\Delta}{T})\,K\,e^{-r(T-t)} \qquad (11)$$

The individual level premium method gives the following contribution:

$$\pi(t) = \pi(0) - rK\Delta\int_0^t \frac{e^{\Delta(t-s)}}{(e^{r(T-s)}-1)}ds \qquad (12)$$

which can be easily computed numerically.

We can also consider the following rate structure (still with a constant benefit K):

Case 2 - jump:

$$\begin{aligned}&\delta(t) = r && \text{for} \quad 0 \le t < t_1 \\ &\delta(t) = r + \Delta && \text{for} \quad t \ge t_1 \\ &(\text{with} \quad \Delta \ne 0)\end{aligned}$$

The unit credit cost method gives the following contribution:

$$\begin{aligned}&\pi(t) = \frac{K}{T} e^{-r(T-t)} && (\,0 \le t < t_1) \\ &\pi(t) = \frac{K}{T} e^{-r(T-t)} (1 - \Delta(t - t_1)) && (\,t \ge t_1)\end{aligned} \tag{13}$$

The individual level premium method gives the following contribution:

$$\begin{aligned}&\pi(t) = \pi(0) && (\,0 \le t < t_1) \\ &\pi(t) = \pi(0) e^{\Delta(t-t_1)} - rK\Delta \int_{t_1}^{t} \frac{e^{\Delta(t-s)}}{(e^{r(T-s)} - 1)} \, ds && (t \ge t_1)\end{aligned} \tag{14}$$

All these formulas can be easily adapted for variable capital; for instance considering a constant difference (case 1) with a variable capital gives the following formula:
- in unit credit we obtain:

$$\pi(t) = \frac{1 + t(g - \Delta)}{T} K e^{gt} \, e^{-r(T-t)} \tag{15}$$

- in individual level premium we obtain:

$$\pi(t) = \pi(0) e^{\Delta t} + rK \int_{0}^{t} \frac{(g - \Delta) e^{gs} e^{\Delta(t-s)}}{e^{r(T-s)} - 1} ds \tag{16}$$

5. ***Numerical illustrations***

In this section we compare the two funding methods (Unit credit and individual level premium) in six different situations. The basic assumptions are the following:

- Maturity: $T=35$
- Actuarial force of interest : $r = \ln(1.04)$

The first three cases are without gains or losses:

Figure 1
Constant benefit: $K=1$ (equations (3) and (4))

Figure 2
Linear benefit growth: $K= 1+0.02\,t$ (equations (7) and (9))

Figure 3
Exponential benefit growth: $K(t) = (1.02)^t$ (formula (6) and (10)).

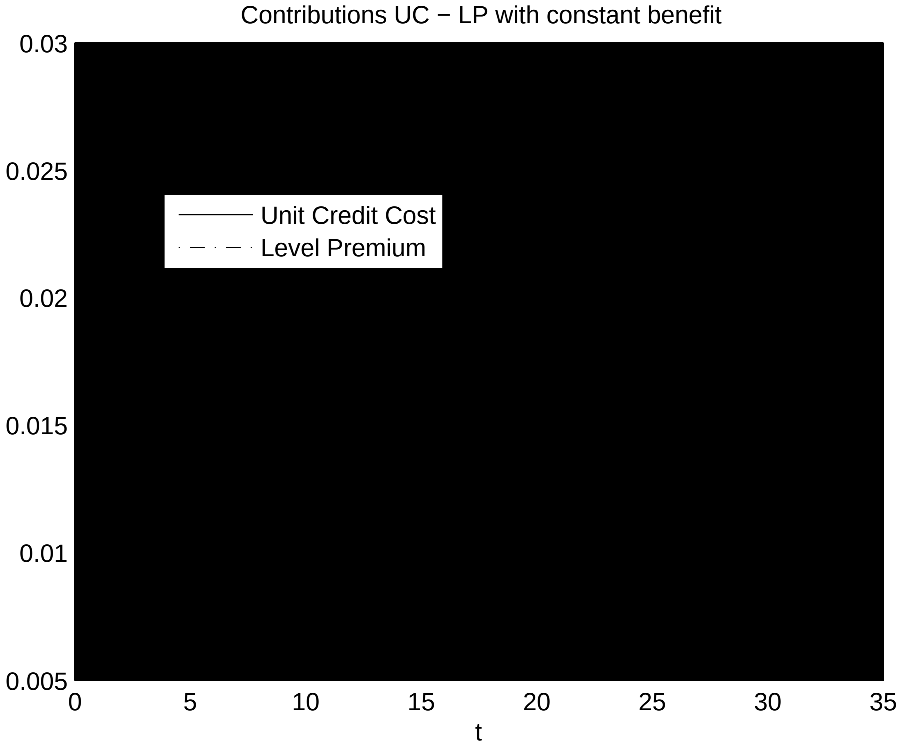
The three last cases are with financial surplus:

Figure 4
Constant benefit; constant difference between the assets return and the discount rate; two situations are considered:
$\delta = \ln(1.03)$ or $\delta = \ln(1.05)$) (equations (11) and (12))

Figure 5
Constant benefit; jump in the assets return at time $t = 20$ (from $\delta = \ln(1.04)$ to $\delta = \ln(1.03)$ or $\delta = \ln(1.05)$) (equations (13) and (14))

Figure 6
Exponential benefit growth as in case 3; constant difference as in case 4 (equations (15) and (16)).

Figure 1:
Constant Benefit : K=1



As expected, in a static environment, the contribution in the individual level premium method is constant, whereas the contribution is growing exponentially in the unit credit cost. But as soon as dynamic evolutions are considered, the effect is just the opposite as shown in the following figures.

Figure 2

Variable Benefit With Linear Growth : K(t) = 1 + 0.02 t

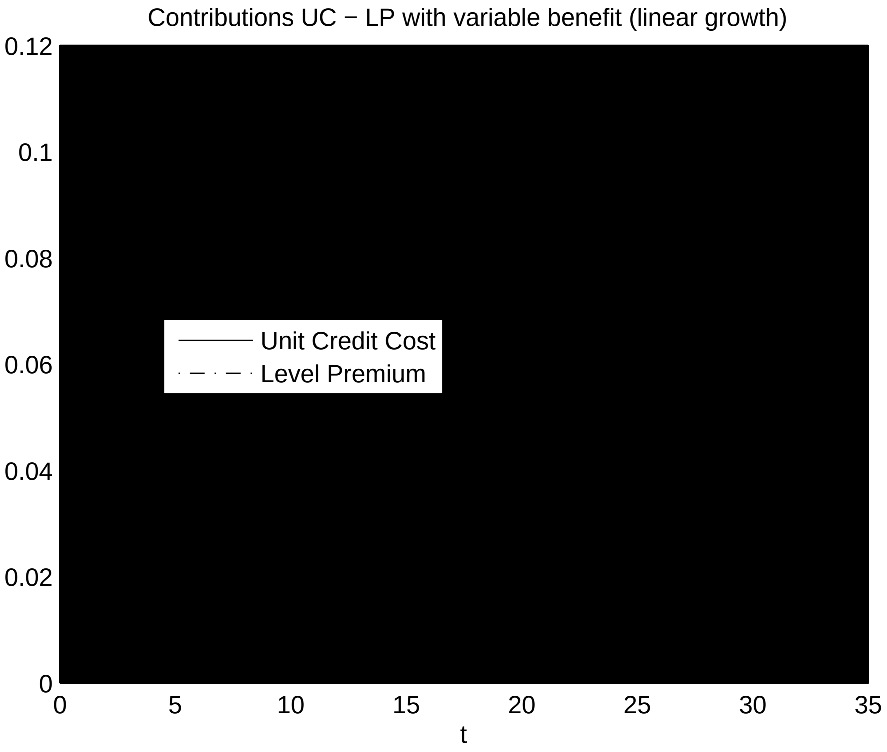


Figure 3

Variable Benefit With Exponential Growth: $K(t) = (1.02)^t$

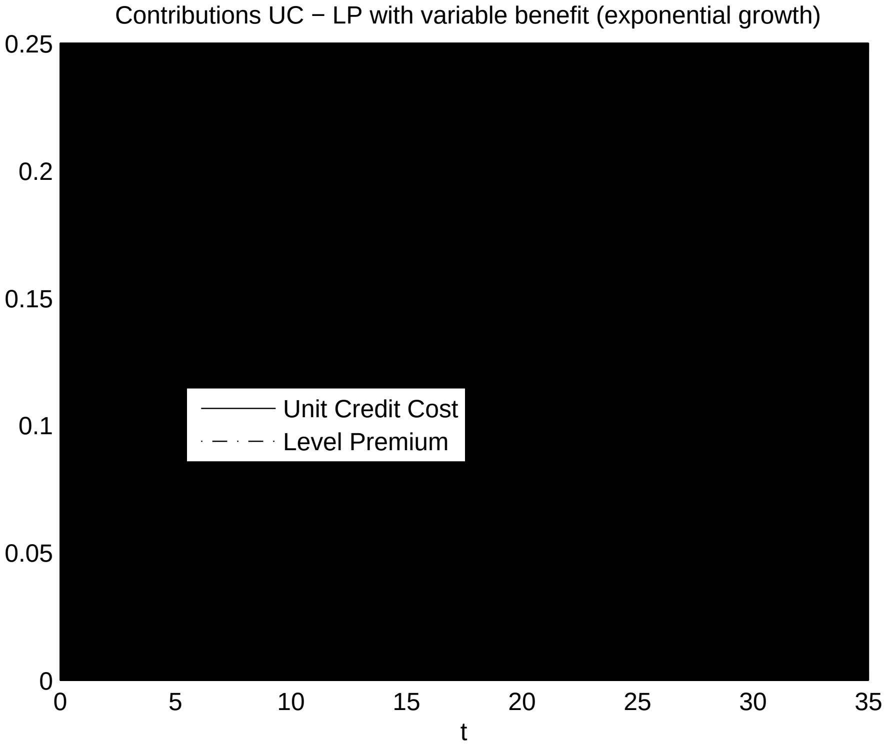


Figure 4
Constant Benefit And Constant Lag

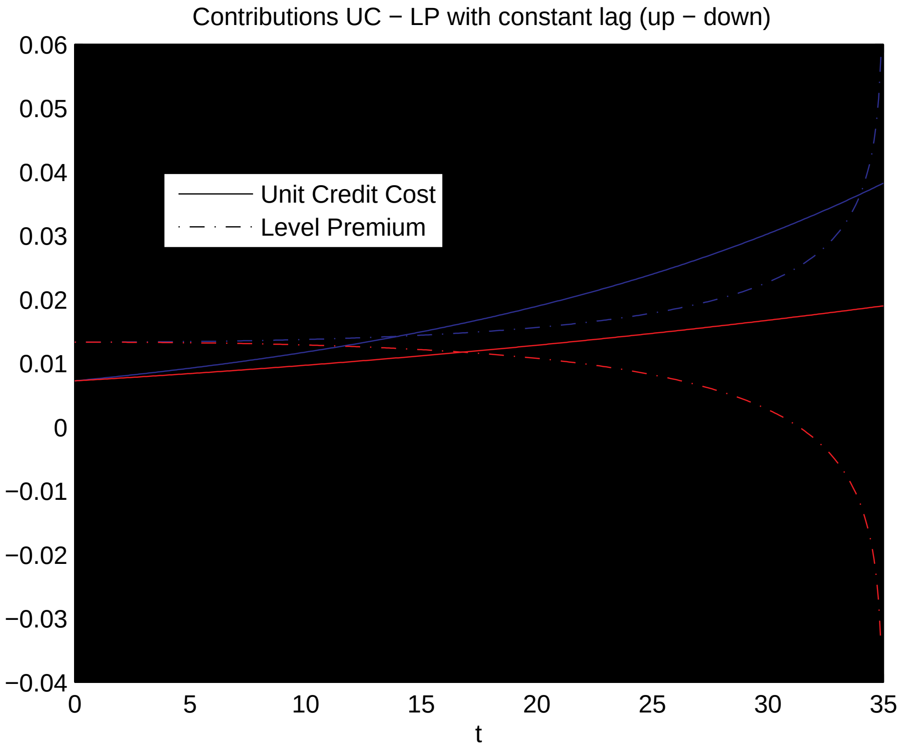


Figure 5
Constant Benefit and Jump

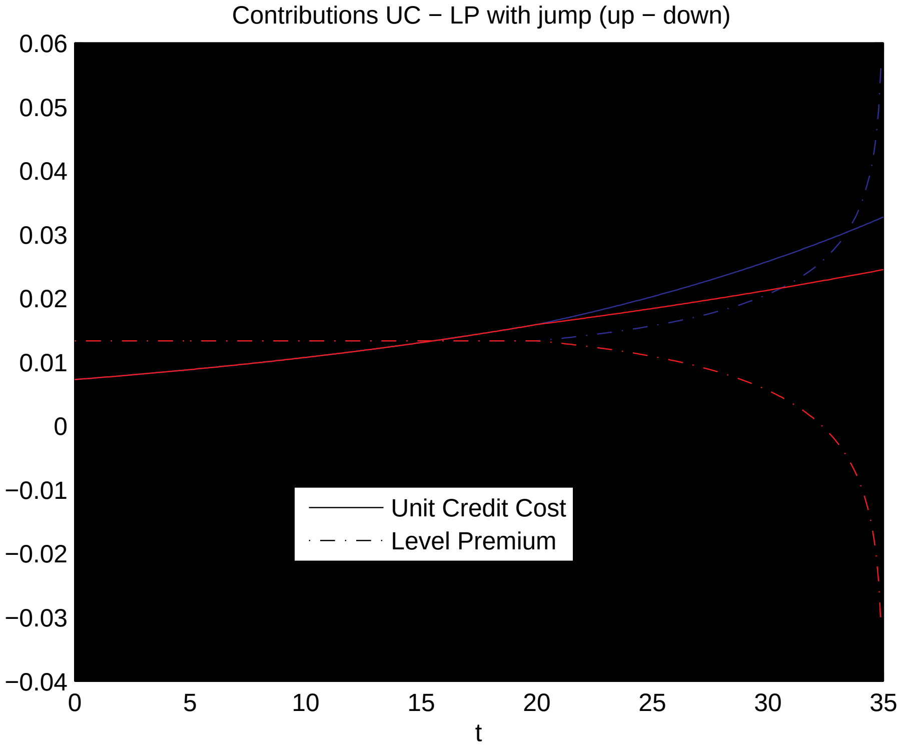
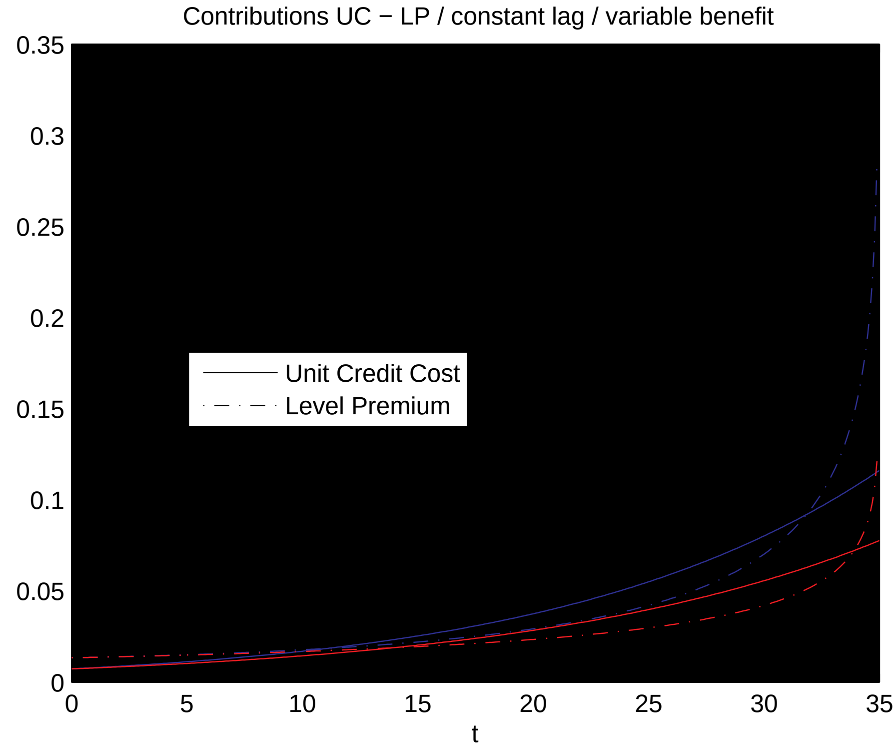


Figure 6

Variable Benefit With Exponential Growth And Constant Lag



6. Conclusion

Calculus has allowed us to obtain easy explicit formulations of contributions and to compare two important funding schemes: the unit credit cost method and the individual level premium method. While the unit credit cost philosophy seems to be safe and coherent with respect to changes in the benefits or in the rate of return on assets, the dangers of the individual level premium method have been highlighted. In particular, in a continuous time deterministic model, contributions in individual level premium are not bounded for changes in benefit or in rate of return closed to the maturity of the contract. Of course in practice, periodic contributions are computed instead of continuous densities but the property of unbounded density leads then to huge increase of regular contributions just before maturity. This fact should definitely convince pension managers to abandon such a method and to go to unit credit cost principles as recommended by the IAS norms.

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