MANAGEMENT OF CATASTROPHIC RISKS CONSIDERING THE EXISTENCE OF EARLY WARNING SYSTEMS

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Abstract

It is of crucial importance to incorporate the notion of early warning systems in insurance mathematics. We develop the concept of an arrival process considering an early warning system and we use it to create actuarial models considering the existence of early warnings.

As an example of application, we formulate a stochastic optimization problem using our model to find an investment strategy for the management of a fund from a governmental perspective and we solve it using the Föllmer-Schweizer strategy.

Keywords: risk management, catastrophic risks, early warning systems
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It is of crucial importance to incorporate the notion of early warning systems in insurance mathematics. We develop the concept of an arrival process considering an early warning system and we use it to create actuarial models considering the existence of early warnings.

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1 INTRODUCTION

Early warning is defined in [11] as a warning arriving in time before an imminent natural hazard. The basic structure of an early warning system has three consecutive phases:

i. forecasting,
ii. warning and
iii. reaction.

(see [11]). In Figure 1, we show the main components of an early warning system in a schematic representation.

The forecasting phase is based on a scientific and technical analysis of data from measurement stations. Its goal is to give a trustworthy forecast of natural phenomena in magnitude, time behaviour and location. If a natural phenomenon, which could have catastrophic consequences is forecasted, a warning is issued.

The top priority of forecasting models is the prediction of the time between the warning and the occurrence of the natural phenomenon ($t_{\text{crit}}$). The precise prediction of the event magnitude and its future progression is in many cases not so important as estimating $t_{\text{crit}}$.

From the predictions of $t_{\text{crit}}$, we can have an idea of how much time we have for the warning and the reaction phases by type of disaster. For example, droughts and volcanic eruptions can be foreseen within months of anticipation; flood warnings usually happen within 24 hours of anticipation, and earthquake predictions within seconds of anticipation (see [11]).
There are a variety of forecasting methodologies. Some of them are essentially deterministic (e.g. forecasting volcanic eruptions) and others are based on the extrapolation of a continuous time-dependent process (e.g. flood forecasting).

During the warning phase, information is circulated. The success of this phase depends on technical, social and political factors. Political problems can arise when deciding if endangered populations should be warned. Miscommunication between forecasting experts and authorities should not be underestimated, as there can be serious consequences, like in the case of the volcanic eruption of the Nevado del Ruiz, Colombia, in 1985. The warning of geologists was not taken seriously and approximately 25000 people died (see [11]).

Important decision criteria to spread a warning or not is the margin of error of the forecast, the possible consequences of the natural event and the economic cost of the reaction phase. Actually, it is difficult for decision makers to evaluate the benefits of spreading a warning. Too many false alarms can diminish the reliability of official announcements.

If authorities decide to spread the warning, endangered populations should be informed as quickly as possible. Information and communication technology can facilitate this task; but however, in some countries there are regions without access to these developments.

The reaction phase is dominated by sociological factors. This is the time to carry out emergency plans. In many countries, organizational and administrative problems often impede a working emergency management. Other problems, often underestimated, are distortions in the risk perception of natural disasters, ignorance about how to react in case of an emergency, difficulties to understand what experts are trying to communicate, and mistrust in the authorities (civil, military or governmental). In addition, poverty, marginalization, war, delinquency, bands of guerrillas, terrorism and epidemics can also complicate the emergency management.

We can also apply the above-mentioned concepts for the case of man-made hazards that sometimes are warned-events, like terrorism. The forecasting phase can be considered as being equivalent to a investigation-phase. However, it should be notice that not every warning phase is preceded by a investigation-phase. The origin of the alert can be an external source.

Now, with illustration purposes, we explain the basic ideas of earthquake early warning systems.

1.1 EARTHQUAKE EARLY WARNING SYSTEMS

In spite of the short time period for the warning and the reaction phases, in some cases it is possible to take actions for loss prevention. For example: the shutdown of computers, disk drives, high precision facilities, airport operations, electronic facilities, high energy facilities, gas distribution, refineries, nuclear power plants and water pipelines; the rerouting of electrical power; stopping or slowing trains and subways; alerting hospital operating rooms; opening fire station doors; starting emergency detonators; leaving elevators in a safe position; shutting off of oil pipelines; issuing audio alarms, and moving to a safe state in nuclear facilities (see [15]). Some of these measures have been implemented or are under
consideration in Japan, Mexico, Taiwan, California, Romania, Turkey and other countries ([15]).

Mexico has a public earthquake early warning system ([10]). It was implemented because was foreseen that an earthquake of magnitude greater than M7, with epicenter in the coasts of Guerrero, between Acapulco and Zihuatanejo, could occur. This event would have catastrophic consequences for Mexico City.

The Mexican Seismic Alert System (SAS) transmits a warning to Mexico City if an earthquake is forecasted at the above mentioned coast segment. An alert is technically possible, because the warning issued in Guerrero is transmitted to different points in Mexico City using electromagnetic waves and they travel faster than seismic waves. The distance between the Guerrero subduction zone and Mexico City is about 320 km. Therefore, the warning time lasts about 60-70 seconds ([1], [10]).

Figure 1: Diagram of a warning system. Source: [11]

The warning is spread out automatically depending on the magnitude of the forecasted earthquake:
I. Public alert: seismic energy, measured at the beginning of an undergoing earthquake, produces a forecast for an earthquake of magnitude greater than M6.
II. Preventive or restricted alert: seismic energy, measured at the beginning of an undergoing earthquake, produces a forecast for an earthquake of magnitude below M6.

Public alerts are mainly transmitted by radio and television. The alarm is also transmitted to schools, the subway and some buildings. Take into consideration that the number of alerted people varies depending on the time of the day in which the alert is issued.
To conclude this section, we wish to stress that the success of every early warning system not only depends on a good forecasting, but also on the capacity to reach authorities and endangered population as soon as possible, and on the decisions following the alert.

2 QUANTIFYING THE ECONOMIC IMPACT OF EARLY WARNING SYSTEMS

Within the last years, substantial progress in early warning systems for some types of natural hazards has been reached. These models are subject to constant improvement and are already fundamental for damage reduction. It is of crucial importance to incorporate these advances into insurance mathematics. This research presents a first step towards this direction.

In some countries, private and public sectors share natural disaster risks. However, to model this partnership mathematically, we need models for both parts. With this motivation, we developed a model for the reserve of an insurance company and for a governmental fund for natural disasters considering an early warning system. We conceived simplified arrival processes for expenditures (governmental fund) and claims (reinsurance company) using the information that we identified as fundamental.

As an example of application, we use our model to find an optimal management strategy for the above mentioned governmental fund using the Föllmer-Schweizer strategy.

2.1 THE INFORMATION PROBLEM

The first step was finding out what information regarding warning systems is essential for our goals. Observing that the natural hazard's forecasting and anticipating economic needs under uncertainty are two problems of different nature.

As we explained above, the natural hazard’s forecasting intends to predict the occurrence of it in the near future. For economic planning under uncertainty, we need mathematical models to describe economic damage due to natural disasters stochastically. We also wish to stress that not all natural hazards become natural disasters.

We identify two main arrival processes involved in our problem: the early warnings (ex-ante) and the claims (ex-post). The main challenge for the development of a parsimonious mathematical model was that both processes are dependent and not simultaneous.

We explored different possibilities for the modelling, some of them focused on $t_{\text{crit}}$ and the dependence between the arrivals of warnings and natural disasters. We modelled the warning system as a continuous stochastic model that should generate different cases: false warning, non-warned disaster, not enough time to react, the error $t_{\text{crit}}-t_{\text{obs}}$ where $t_{\text{obs}}$ is the observed time between the warning and the disaster. Additionally, we explored the convenience of modelling the economic benefit of a hitting warning as a separate random variable.
As a main result, we conceived a simplified model that captures the essence of our problem with a minimum of information. We realized that, if \( t_{crit} \) is small enough, we can exploit performance statistics of early warning systems for the actuarial modelling.

The main idea is to take advantage of the only fact we have for sure: every warning system is fallible. From this axiom, we deduce that we have essentially three different types of errors:

- **Type I**: a warning was issued and nothing happened (\( P[\text{Error type I}] = \alpha_1 \)),
- **Type II**: a disaster occurred and no warning was issued (\( P[\text{Error type II}] = \alpha_2 \)),
- **Type III**: a warning was issued and a disaster occurred, but the system failed in the warning or the reaction (\( P[\text{Error type III}] = \alpha_3 \)).

In order to estimate the probability of an error type III (\( \alpha_3 \)) from data, we require the development of standard criteria for the classification of events in order to generate suitable data bases. This is a topic for further research in sociology. Up to now, data bases for natural disasters are very imprecise and there are no standards to structure them.

We can estimate \( \alpha_1 \) and \( \alpha_2 \) from statistical information. If we don't have enough information available or if the information is too imprecise, we recommend generating synthetic data bases.

### 3 ARRIVAL PROCESS OF DISASTERS

Assume that the arrival process of claims corresponding to the type of disaster of our interest \( \{N(t), t \geq 0\} \) is compound Poisson with rate \( \lambda > 0 \). Then the inter-arrival times are given by the sequence of independent identically distributed random variables (i.i.d.r.v.) \( (T_i), i=1,2,\ldots, \).

The probability of \( n \) claims within the time interval \((0,t]\) is \( P(N(t) = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!} \) and \( E[N(t)]=\lambda t \).

The sequence of i.i.d.r.v. \( (T_i)_{i \geq 1}, T_i \sim \exp(\lambda) \) is the sequence of inter-arrival times of \( N(t) \). Let \( (Z_i)_{i \geq 0} \) be the sequence of arrival times \( Z_n = \sum_{i=1}^{n} T_i \). In other terms, \( Z_n = \inf\{t \geq 0, N(t)=n\} \).

The probability density of \( Z_n \) is given by

\[
f_{Z_n}(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}, \quad t \geq 0. \tag{1}
\]

Then, \( Z_n \sim \Gamma(n,\lambda) \) and \( E[Z_n]=n\lambda^{-1} \).

The number of claims from warned disasters in \((0,t]\) is \( N^*(t) = \sum_{i=1}^{N(t)} U_i \), where \( U_i \sim \text{Bern}(1-\alpha_2) \) is defined by

\[
U_i = \begin{cases} 
1 & \text{if a warning was issued for the disaster that originated the claim,} \\
0 & \text{otherwise.} 
\end{cases} \tag{2}
\]

It is easy to verify that the arrival process of claims from warned disasters \( (N^*(t)) \) is compound Poisson with rate \( \lambda^* = (1-\alpha_2) \lambda \), inter-arrival times \( T^r_i \sim \exp(\lambda^*) \) and arrival times -
The arrival process of non-warned disasters (\( \hat{\mathcal{N}}(t) \)) is compound Poisson with rate \( \tilde{\lambda} = \alpha_3 \lambda \), inter-arrival times \( \tilde{T}_i \sim \exp (\tilde{\lambda}) \) and arrival times \( \hat{Z}_n \sim \Gamma(n, \lambda^*) \). Furthermore, \( \mathcal{N}'(t) \) and \( \hat{\mathcal{N}}(t) \) are independent processes and \( \mathcal{N}(t) = \mathcal{N}'(t) + \hat{\mathcal{N}}(t) \). This is the so-called disaggregation property of Poisson processes.

Finally, we consider the error type III. Using again the disaggregation property of Poisson processes, we decompose the arrival process of warned disasters into two independent Poisson processes. So, we represent the process of warned disasters \( \mathcal{N}'(t) = \hat{\mathcal{N}}(t) + \hat{\mathcal{N}}(t) \) as the sum of the counting process of disasters with non-working warning (\( \hat{\mathcal{N}}(t) \)) and disasters with working warnings (\( \hat{\mathcal{N}}(t) \)). \( \hat{\mathcal{N}}(t) \) and \( \hat{\mathcal{N}}(t) \) are Poisson processes with parameters \( \hat{\lambda} = \alpha_3 \lambda^* = \alpha_3 (1-\alpha_2) \lambda \) and \( \hat{\lambda} = (1-\alpha_2) \lambda^* = (1-\alpha_2)(1-\alpha_3) \lambda \), respectively.

We define the arrival process \( \overline{\mathcal{N}}(t) \) as the sum of the arrivals of non-warned disasters (\( \hat{\mathcal{N}}(t) \)) and disasters preceded by a non-working warning (\( \hat{\mathcal{N}}(t) \)). That is, \( \overline{\mathcal{N}}(t) = \hat{\mathcal{N}}(t) + \hat{\mathcal{N}}(t) \). It is easy to verify that the arrival process of disasters is the sum of \( \mathcal{N}(t) = \overline{\mathcal{N}}(t) + \hat{\mathcal{N}}(t) \).

**4 ARRIVAL PROCESS OF WARNINGS**

Let the arrival process of warnings, \( \{ \kappa(t), t \geq 0 \} \), be a Poisson process with fixed parameter \( \sigma \). From now on, we will denominate emergency to the phases following the warning. Let \( T_i \) be the duration of the \( i \)-th emergency. We can consider the duration of the emergencies \( T_i \) as i.i.d.r.v., \( T_i \sim \text{Erlang}(k, \mu) \), \( i=1,2, \ldots \). It means,

\[
f_{T_i}(t) = \frac{(\mu k)^k}{(k-1)!} t^{k-1} e^{-\mu t}
\]

for \( t \geq 0 \), where \( \mu > 0 \) and \( k \in \mathbb{N} \). We have \( E[T_i] = \mu^i \) and \( \sqrt{\text{Var}[T_i]} = (\sqrt{k} \mu)^{-1} \). \( k \) specifies the degree of variability and is known as the shape parameter.

Note that \( \mu \) is related with the mean duration of the emergency. That is, \( h = \mu^{-1} \). Then, for a 72-hours warning \( h = 72(24)^{-1}(365)^{-1} \approx 0.00822 \) and \( \mu = 121.67 \).

For flooding in Germany, e.g., forecasts succeed between 12 and 48 hours before the event and we usually have 12-hours warnings ([4]).

The parameter \( \sigma \) in our model is associated with the number of warnings expected per year. If, e.g., \( 4 \) warnings are expected, then \( \sigma = 4 \).

We suggest an Erlang distribution for the random variables \( T_i \), because it allows a variance equal to or smaller than the mean \( \mu^{-1} \). If desired, it is possible to model the arrivals of warnings and disasters considering a queueing model of the type \( M/\text{E}_{k}/s \) with \( T_i \) as the “service times”. Note that if we set \( k=1 \), then we have the usual queueing model with exponential “service times”. An emphasis on the queueing model is, however, beyond the scope of this article.

By the disaggregation property of Poisson processes, we separate the arrival process of warnings \( \kappa(t) \) into two independent Poisson processes: \( \kappa^*(t) \) and \( \kappa^*(t) \). The arrival process of effective warnings, \( \kappa^*(t) \), has parameter \( \sigma^* = (1-\alpha_1) \sigma \) and the arrival process of false warnings, \( \kappa^*(t) \), has parameter \( \sigma^* = \alpha_1 \sigma \).
Note that the number of warned disasters and the number of technically-successful warnings should be the same. Hence, $\kappa^*(t)$ and $\hat{N}(t)$ are Poisson processes with the same intensity. Using the equality $\sigma^*=\lambda^*$, we obtain an expression for $\sigma$ in terms of $\lambda$:

$$\sigma = \frac{1-a_2}{1-a_j} \lambda.$$  \hfill (4)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Equivalence</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda^*$</td>
<td>$(1-a_2)\lambda$</td>
</tr>
<tr>
<td>$\lambda^-$</td>
<td>$a_2\lambda$</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>$a_3(1-a_2)\lambda$</td>
</tr>
<tr>
<td>$\hat{\lambda}$</td>
<td>$(1-a_3)(1-a_2)\lambda$</td>
</tr>
<tr>
<td>$\tilde{\lambda}$</td>
<td>$(a_2+a_3-a_2a_3)\lambda$</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>$(1-a_2)(1-a_1)\lambda$</td>
</tr>
<tr>
<td>$\sigma^-$</td>
<td>$a_1(1-a_2)(1-a_1)\lambda$</td>
</tr>
<tr>
<td>$\eta$</td>
<td>$(1+a_1(1-a_2)(1-a_1))\lambda$</td>
</tr>
</tbody>
</table>

Table 1: Poisson parameters in terms of the mean number of disasters per year ($\lambda$) and the probabilities of error ($a_1$, $a_2$, $a_3$).

### 5 ARRIVAL PROCESS OF EXPENDITURES AND CLAIMS

In the previous sections, we decomposed the arrival processes $N(t)$ (disasters) and $\kappa(t)$ (warnings) into independent processes. A summary of all parameters in terms of $a_i$, $a_2$, $a_3$ and $\lambda$ is given in Table 1.

For the stochastic modelling, we consider a simplified arrival process of expenditures (or claims). We model the arrival process of expenditures using a Poisson process $\{J(t), t \geq 0\}$ with parameter $\eta$, where

$$\eta = \lambda + \tilde{\lambda} + \sigma^-.$$  \hfill (5)

That is,

$$\eta = \lambda + \tilde{\lambda} + \lambda^- + \sigma^-.$$  \hfill (6)

The process of technically-successful warnings, $\kappa^*(t)$, is implicit in Eqs. (5) and (6).

We distinguish between ex-post aggregate claims of disasters which occurred after an optimal ex-ante disaster management, and ex-post aggregate claims which have a different background by using different stochastic models, respectively. We do not model any relationship between the size of the losses and the frequency of the extreme event.
The classical Lundberg model for the risk reserve (capital) of an insurance company is given by
\[ R_t = x + ct - S_t, \quad t \geq 0, \]
where
\[ S(t) = \sum_{i=1}^{N(t)} S_i \]
denotes the aggregate claims process up to time \( t \), \( x \) is the initial capital, \( c \) is the instantaneous premium rate and \( N(t) \) is the arrival process of the claims. The individual claims \( S_i, \quad i=1,\ldots,N(t) \), are i.i.d.r.v.. The Lundberg model is widely used in practice and is the basis for several fundamental results in actuarial mathematics. Nevertheless, this model was conceived before the existence of early warning systems, so that it does not consider the possibility of loss mitigation.

We denote \( \hat{S}(t) \) to the aggregate claims up to time \( t \) from disasters that were preceded by a working emergency management and \( \bar{S}(t) := S(t) - \hat{S}(t) \) are the rest of the claims. If early warning systems have an effect in the claim sizes, this should be reflected in the probability distribution. For this reason, we classify claims according to the alert phase.

Define the aggregated claim process
\[ \hat{S}(t) = \sum_{i=1}^{\hat{N}(t)} \hat{S}_i, \quad \hat{S}_i, \quad i=1,\ldots, \hat{N}(t), \quad \text{i.i.d.r.v.} \]
\[ \bar{S}(t) \text{ is defined in analogy to Eq. (9).} \]

As a result, we obtain a variant of the classical Lundberg model considering the effect of a warning system:
\[ R_t = x + ct - \hat{S}(t) - \tilde{S}(t), \quad t \geq 0. \] (10)

The expectation of the total aggregate claims up to time \( t \geq 0 \) is
\[ \mathbb{E} [ \hat{S}(t) + \tilde{S}(t) ] = (\hat{\lambda} \mathbb{E} [ \hat{S} ] + \bar{\lambda} \mathbb{E} [ \tilde{S} ] ) t. \] (11)

The variance is given by
\[ \text{Var} [ \hat{S}(t) + \tilde{S}(t) ] = (\hat{\lambda} \mathbb{E} [ \hat{S}^2 ] + \bar{\lambda} \mathbb{E} [ \tilde{S}^2 ] ) t. \] (12)

The first difference between the Lundberg model and our model, is that we classify the aggregate claims \( S(t) \) in two types and we consider them as two different processes for the stochastic modelling. Observe that if \( \hat{S}(t) \) and \( \tilde{S}(t) \) are equally distributed, we have the Lundberg model. As we remember, \( \hat{N}(t) \) and \( \tilde{N}(t) \) are compound Poisson processes with rates \( \hat{\lambda} = (1 - \alpha_3)(1 - \alpha_2)\lambda \) and \( \bar{\lambda} = (\alpha_3 + \alpha_2 - \alpha_2 \alpha_3)\lambda \). The parameter of \( N(t) \) is \( \lambda = \hat{\lambda} + \bar{\lambda} \). From the independency of the arrival processes \( \hat{N}(t) \) and \( \tilde{N}(t) \), the aggregated claim processes \( \hat{S}(t) \) and \( \tilde{S}(t) \) are also independent.

However, the most important difference is that, by construction, this variant makes it possible to quantify and compare the impact in the risk reserve of early warning systems with different performance indicators.

### 7 GOVERNMENTAL FUND’S MODEL

The lack of actuarial models from a governmental perspective forced us to look for a real governmental fund for natural disasters to develop our model. Based on an analysis of Mexico’s Fund for Natural Disasters (FONDEN) functioning and history, we conceived the actuarial model for a governmental fund for natural disasters here presented. Our model, however, can also be useful for other countries.

#### 7.1 EXPENDITURE PROCESS

A special feature of Mexico's Fund for Natural Disasters is that it not only considers damage ex-post but also resources for prevention. For this reason, we concluded that we should include two types of expenditures in our mathematical model: ex-ante and ex-post.

Ex-ante expenditures include only resources assigned during alert stages. For example, money provided for temporary shelters after a hurricane warning. A fund’s disbursement after a natural disaster is an ex-post expenditure.

After issuing a warning, whatever the result, we disburse a constant amount \( a_1 \). In case of a false alarm, the quantity \( a_2 \) should be returned to the fund at the end of the emergency phase.
Of course, in the real world $a_1$ is not constant and $a_2$ also varies. Nevertheless, the size of ex-ante expenditures is much easier to calculate than in the ex-post case. In practice, we can use mean observed values to calculate the constants $a_1$ and $a_2$ that we will use. The results should be adjusted according to the experience regarding the variability of ex-ante expenditures.

In our model, expenditures of size $a_1$ have the same arrival process as the warnings: $\kappa(t)$. The reimbursement $a_2$ has the same arrival process as false warnings $\kappa'(t)$. Both processes are dependent, but not the arrival processes of false warnings ($\kappa'(t)$) and working warnings ($\kappa(t)$). Hence, it is mathematically more convenient to work with the arrival processes of the disbursements $a_1$ and $a_1-a_2$ than with the arrival processes of $a_1$ and $a_2$.

<table>
<thead>
<tr>
<th>Event</th>
<th>Expenditure</th>
<th>Arrival process</th>
<th>Poisson parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Technically-successful warning</td>
<td>$a_1$</td>
<td>$\kappa^*(s)$</td>
<td>$\sigma^*$</td>
</tr>
<tr>
<td>False warning</td>
<td>$a_1-a_2$</td>
<td>$\kappa'(s)$</td>
<td>$\sigma^*$</td>
</tr>
<tr>
<td>Claim ex-post (optimal emergency management)</td>
<td>$\tilde{S}_i$</td>
<td>$\tilde{N}(s)$</td>
<td>$\lambda$</td>
</tr>
<tr>
<td>Claim ex-post (deficient emergency management or non-warned disaster)</td>
<td>$\tilde{S}_i$</td>
<td>$\tilde{N}(s)$</td>
<td>$\lambda$</td>
</tr>
</tbody>
</table>

Table 2: Classification of expenditures.

A summary of results is in Table 2.
7.2 FUND’S MODEL

Assume that we have an unlimited XL-reinsurance contract for year \( n \) with fixed retention level \( \alpha^{(n)}>0 \). If there is no reinsurance contract this year, then \( \alpha^{(n)} = \infty \).

Consider that the reinsurer uses the expected value principle with safety loading \( \theta > 0 \) for premium calculation. In this case,
\[
C^{(n)} = (1 + \theta) \left[ E \left( \sum_{i=1}^{N(I)} (\hat{S}_i - \alpha^{(n)})^+ \right) + E \left( \sum_{i=1}^{\bar{N}(I)} (\overline{S}_i - \alpha^{(n)})^+ \right) \right],
\]
(13)

\( \alpha^{(n)} < \infty \). \( C^{(n)} \) is the annual premium to be paid at the beginning of the year \( n \). If \( x_0^{(n)} \) are the initial resources for the year \( n \), it is clear that only contracts with \( C^{(n)} < x_0^{(n)} \) are viable, \( x_0^{(n)} \) is the sum of the budget for year \( n \) and the surplus (if not zero) of the previous year.

In the above formula, we show that it is possible to quantify the effect of risk reduction in the risk premium. A risk premium calculated considering the effects of warnings can represent an economic incentive for risk-averse governments to invest in early warning systems and purchase more reinsurance.

Let the process \( (R^{(n)}(s))_{s \in [0,1]} \) be the level of the fund at time \( t=s+n-I, \ n \in \mathbb{N} \). The initial value of \( (R^{(n)}(s))_{s \in [0,1]} \) is \( b_0^{(n)} = x_0^{(n)} - C^{(n)} \).

The model for the governmental fund at time \( s \) for the \( n \)-year is
\[
R^{(n)}(s) = b_0^{(n)} - \Pi(s),
\]
(14)
where
\[
\Pi(s) = \sum_{i=1}^{\hat{N}(s)} (a_j + \min \{\alpha^{(n)}, \hat{S}_i\}) + \sum_{i=1}^{\bar{N}(s)} (a_j + \min \{\alpha^{(n)}, \overline{S}_i\}) + \sum_{i=1}^{N(s)} \min \{\alpha^{(n)}, \overline{S}_i\} - (a_1-a_2)\kappa^{-}(s).
\]
(15)
\( \Pi(s) \) is a random variable that represents the aggregated expenditures up to time \( s \). The process \( R^{(n)}(s) \) considers ex-ante and ex-post economic resources together. As we mentioned before, in Mexico ex-ante and ex-post economic resources are administrated independently. We joined them for the mathematical formulation because they share the same initial budget.

The expectation and the variance of \( \Pi(s) \) are given by
\[
E[\Pi(s)] = (1-\alpha_j)\lambda s \left[ \frac{a_j - a_i \alpha_j}{1-\alpha_j} + (1-\alpha_j)E[\min \{\alpha^{(n)}, \hat{S}_i\}] \right] + (\alpha_2 + \alpha_j (1-\alpha_j))\lambda s E[\min \{\alpha^{(n)}, \overline{S}_i\}],
\]
(16)
and
\[
\text{Var}[\Pi(s)] = \left( a_2^2 + (a_1-a_2)^2 \frac{\alpha_j}{1-\alpha_j} \right) (1-\alpha_2)\lambda s + \left( (1-\alpha_j)E[\min \{\alpha^{(n)}, \hat{S}_i\}] \right)^2 (1-\alpha_2)\lambda s
\]
\[
+ \left( \alpha_2 + \alpha_j (1-\alpha_j) \right) E[\left( \min \{\alpha^{(n)}, \overline{S}_i\} \right)^2],
\]
(17)
respectively.
For the case $\alpha = \infty$ (no reinsurance), we have

$$E[\Pi(s)] = (\hat{\lambda}E[\hat{S}_i] + \bar{\lambda}E[\bar{S}_i]) + a_i \lambda^\ast s + (a_i - a_j) \sigma^\ast s$$

and

$$\text{Var}[\Pi(s)] = (\hat{\lambda}E[\hat{S}_i^2] + \bar{\lambda}E[\bar{S}_i^2]) + a_i^2 \lambda^\ast + (a_i - a_j)^2 \sigma^\ast s.$$  \hspace{1cm} (18)

We will use the model for the cumulated expenditures $\Pi(s)$ considering an XL-reinsurance contract in order to find an optimal management strategy.

### 8 GENERALIZED-PARETO CLAIMS

In this section, we make some calculations assuming Generalized-Pareto Claims (see Definition 8.1). Finally, we present a numerical example.

Let us first introduce the definition of Generalized Pareto Distribution:

**Definition 8.1 (Generalized Pareto Distribution ([8]))**

Define the density function (df) $G_{\xi,\beta}$, $\xi, \beta \in \mathbb{R}$, $\beta > 0$, by

$$G_{\xi,\beta}(x) = \begin{cases} 
1 - \left(1 + \frac{x - \nu}{\beta} \right)^{-\frac{1}{\xi}} & \text{if } \xi \neq 0, \\
1 - \exp\left\{-\frac{x - \nu}{\beta}\right\} & \text{if } \xi = 0.
\end{cases}$$  \hspace{1cm} (20)

where

$$x \geq \nu \quad \text{if} \quad \xi \geq 0,$$

$$x \leq \nu \quad \text{if} \quad \xi < 0.$$

is called a standard generalized Pareto distribution (GPD).

If $\hat{\xi} = 0$ or $\bar{\xi} = 0$, the respective claims ($\hat{S}_i$ and $\bar{S}_i$) have exponential distribution. For this reason, we restrict ourselves to the sub-exponential case, i.e. $\hat{\xi} \neq 0$, $\bar{\xi} \neq 0$. Additionally, given that the claims ex-post, $\hat{S}_i$ and $\bar{S}_i$, have values in $(0, \infty)$ and finite mean and variance, we set $\hat{S}_i \sim G_{\hat{\xi},\hat{\beta}}$, $\bar{S}_i \sim G_{\bar{\xi},\bar{\beta}}$ with $\hat{\xi}, \bar{\xi} \in \left[0, \frac{1}{2}\right]$. In order to simplify the output, we consider $\nu = 0$ and $\nu = 0$. We can easily generalize the results to other cases.

We have

$$E[\hat{S}_i] = \frac{\hat{\beta}}{1 - \hat{\xi}},$$

$$E[\hat{S}_i^2] = \frac{2\hat{\beta}^2}{(1 - \hat{\xi})(1 - 2\hat{\xi})}$$

and

$$\text{Var}[\hat{S}_i] = \frac{\hat{\beta}^2}{(1 - \hat{\xi})^2(1 - 2\hat{\xi})}.$$
\[ E[\bar{S}_i], E[\bar{S}_i^2] \text{ and } \text{Var}[\bar{S}_i] \text{ are defined in analogy to the three above equations.} \]

The XL-premium (see Eq. (13)) that the government should pay is in this case
\[ C^{(n)} = (I + \theta) \left( \lambda \frac{\hat{\beta} + \hat{\xi} \alpha^{(n)}}{1 - \hat{\xi}} \left( I + \frac{\hat{\xi} \alpha^{(n)}}{\hat{\beta}} \right) \right)^{1/\hat{\xi}} + \frac{\lambda}{I - \hat{\xi}} \left( I + \frac{\hat{\xi} \alpha^{(n)}}{\hat{\beta}} \right)^{-1} \left( I + \frac{\hat{\xi} \alpha^{(n)}}{\hat{\beta}} \right)^{-1/\hat{\xi}}, \] (21)

where \( \alpha^{(n)} \) is the retention level.

The mean ex-post expenditure per disaster, considering an XL-reinsurance, is
\[ E[\min\{\alpha^{(n)}, \bar{S}_i\}] = \frac{\hat{\beta}}{I - \hat{\xi}} - \frac{\hat{\beta} \cdot \hat{\xi} \alpha^{(n)}}{I - \hat{\xi}} \left( I + \frac{\hat{\xi} \alpha^{(n)}}{\hat{\beta}} \right)^{1/\hat{\xi}}. \] (22)

The second moment of the expenditure considering an XL-reinsurance is
\[ E[(\min\{\alpha^{(n)}, \bar{S}_i\})^2] = -2 \left( \frac{\alpha^{(n)} \cdot (\hat{\beta} + \hat{\xi} \alpha^{(n)})}{I - \hat{\xi}} + \frac{(\hat{\beta} + \hat{\xi} \alpha^{(n)})^2}{(1 - \hat{\xi})(1 - 2\hat{\xi})} \right) \left( I + \frac{\hat{\xi} \alpha^{(n)}}{\hat{\beta}} \right)^{1/\hat{\xi}} \]
\[ + \frac{2\hat{\beta}^2}{(1 - \hat{\xi})(1 - 2\hat{\xi})}. \] (23)

\[ E[\min\{\alpha^{(n)}, \bar{S}_i\}] \text{ and } E[(\min\{\alpha^{(n)}, \bar{S}_i\})^2] \text{ are defined in analogy to Eqs. (22) and (23).} \]

Substituting the equations for the mean and the second moment of the expenditures considering an XL-reinsurance contract in Eqs. (16) and (17), we obtain formulas for the calculation of the mean and the variance of the total aggregated outcome up to time \( s \).

### 8.1 Numerical Example

Let the number of periods be \( n=1 \). Assume we have a warning system with performance indicators \( \alpha_1=0.2 \) (error type I), \( \alpha_2=0.3 \) (error type II) and \( \alpha_3=0.4 \) (error type III). We will make an example based on the assumption that the planning is made considering a mean of 5 disasters per year.

We classify claims accordingly in two groups:
1. a warning was issued and it was technically and sociologically effective,
2. the rest of the disasters.

The next step would be to fit in a distribution for every group of data and to set the corresponding parameters. If the available information is not enough, then we need to make some assumption according to our experience.

Suppose that we adjust a generalized Pareto distribution for both types of claims and we obtain \( \hat{\xi} = 0.3, \hat{\beta} = 50, \zeta = 0.4 \) and \( \hat{\beta} = 100 \). The expenditures and the reimbursement for emergency management are \( \alpha_1=60 \) and \( \alpha_2=40 \), respectively. Consider a XL-reinsurance contract with retention level \( \alpha^{(n)}=1000 \) and safety loading \( \theta=2 \).
Our calculations yield $\eta=5.87$, $\sigma=4.37$, $\hat{\lambda}=2.10$ and $\overline{\lambda}=2.90$. The expectation and variance of $\hat{S}_t$ (optimal emergency management) and $\overline{S}_t$ (deficient emergency management) are $E[\hat{S}_t]=71.43$, $\text{Var}([\hat{S}_t]=12,755.10$, $E[\overline{S}_t]=166.67$ and $\text{Var}([\overline{S}_t]=138,888.89$. The corresponding premium, as defined in Eq. (13), is $C^{(\alpha)}=134.49$.

Moreover, $E[\Pi(I)]=816.00$, $\text{Var}[\Pi(I)]=225,170.00$, $E[\min\{\alpha^{(\alpha)},\hat{S}_t]\}=70.67$ and $E[\min\{\alpha^{(\alpha)},\overline{S}_t]\}=151.76$.

The influence of the warning system should be explicitly reflected in the difference in the results for both types of claims and implicitly in the premium and statistics for the cumulated expenditures at the end of the fiscal year $\Pi(I)$. Note, that if we consider that an early warning system has an effect on the claim sizes, the benefit of an effective risk mitigation is translated into a premium reduction. The more effective the early warning system is, the lower the premium.

The effects of the XL-reinsurance contract can be identified in the statistics of $\Pi(I)$ with and without reinsurance. If there is no reinsurance contract, $E[\Pi(I)]=860.85$ and $\text{Var}[\Pi(I)]=533,783.33$.

Finally, we calculate the mean and the variance of the sum of all claims: $E[\hat{S}(I)+\overline{S}(I)]=633.35$ and $\text{Var}[\hat{S}(I)+\overline{S}(I)]=520,833.33$.

Certainly, we can also make a variety of comparative analysis changing the XL-contract conditions, the performance indicators, the claim parameters, etc.

In Figures 3 and 2, we illustrate a simulation of the governmental fund and of the risk reserve during 5 years, respectively. With the help of these illustrations, we can appreciate the main differences between both processes. The governmental fund has an income once a year (budget), expenditures and reimbursements (or negative expenditures) for emergency management as well as post-disaster expenditures. The risk-reserve of the insurance company has a continuous income for premiums and claims post-disaster.

In the following sections, as an example of application, we calculate an optimal management strategy for a governmental fund for natural disasters that combines investment and XL-reinsurance.

9 MARKET MODEL

The market model that we use for the optimization problem is the classical Black-Scholes. We have a bank account paying interest rate $r$ and a risky asset $Z$ in which the fund's manager can invest. The model can be generalized to consider a finite number of risky assets. We use the classical Samuelson model for the dynamics of the asset price $Z(s)$:

$$dZ(s)=Z(s)(\mu ds+bdB(s)), \quad Z(0)>0, \mu>0, b>0.$$  

(24)
Let $B(s)$ be a standard Wiener process. $F^{B,\Pi}$ denotes the $P$-augmentation of the filtration generated by $B(s)$ and the process $\Pi(s)$. Consider a probability space $(\Omega, \mathcal{F}, F, P)$ with a filtration $F = (\mathcal{F}_t)_{t \leq s \leq 1} \supseteq F^{B,\Pi}$ satisfying the usual conditions and $\mathcal{F}=\mathcal{F}_I$.

### 10 Market Assumptions

We work in the framework of incomplete markets and no-arbitrage.

Denote by 
\[ M^e(P) = \{ \tilde{P} \equiv P : \tilde{P} \text{ is a probability measure and } Y \text{ is a } \tilde{P} \text{-local martingale} \} \]
the set of all probability measures $\tilde{P}$ on $\mathcal{F}$ which are equivalent to $P$ in $[0,1]$.

If there is no possibility of arbitrage, we can assume $M^e(P) \neq \phi$ ([7]).

Incompleteness implies that we can not replicate the contingent outcome with the instruments available in the market without risk. The total contingent expenditures of the fund $(\Pi(I))$ depend on a source of randomness which does not influence the market's coefficients $\mu, r$ and $b$.

### 11 Investment Portfolio

Our aim is to find whether we can combine a risk free instrument (bank account) with risky investment (asset $Z$) to hedge the contingent total outcome $\Pi(I)$ of the governmental fund for natural disasters.

With development purposes, we assume Pareto-distributed claims. The effect of an XL-reinsurance contract is considered in the model for $\Pi(I)$. In this section, we explain the model for the investment portfolio that we use to explore the possibility of managing risk using investment.

From the reserve available at the beginning of the year $n$ ($R^{(n)}(0)$), we subtract the XL-premium ($C^{(n)}$) and other payments to be made at the beginning of the year. We denote the total amount to be subtracted as $\bar{P}$, where $\bar{P} \leq C^{(n)} \leq 0$. If the reserve suffices (i.e. $R^{(n)}(0) > \bar{P}$), we assign an amount $x \in (0, R^{(n)}(0) - \bar{P})$ for the risk management strategy based on investment.

Define a real-valued process $X^{(n)}(s)$ with $X^{(n)}(0) = x$ representing the current wealth at time $t=s+n$. Let $\pi(s)$ be the number of shares $Z(s)$ held at time $t$. At each point in time $t$, the amount $\pi(s)Z(s)$ is invested into the risky asset. The difference $X^{(n)}(s) - \pi(s)Z(s)$ is left in a bank account earning interest $r$. We deduce that the dynamics of the investment portfolio $X^{(n)}(s)$ during the year $n$ is given by
\[
dX^{(n)}(s) = rX^{(n)}(s)ds - r\pi(s)Z(s)ds - r\pi(s)Z(s)ds + \pi(s)dZ(s),
\]
\[ X^{(n)}(0) = x, \quad x \geq 0, \quad 0 \leq s \leq 1. \]
Rewriting Eq. (25),
\[ dX^{(n)}(s) = rX^{(n)}(s)ds + \pi(s)Z(s)((\mu - r)ds + dB(s)), \]
where \( B(s) \) is a standard linear Brownian Motion.

All processes here considered are indexed in \([0,1]\). We calculate the optimal strategy for a time-horizon of one year. In order to simplify the notation, from now on we will write \( X(s) \) instead of \( X^{(n)}(s) \).

For our purposes, it is mathematically convenient to work with the discounted prices and portfolio values and to make a change of measure that eliminates the drift term \((\mu-r)ds\) in Eq. (26) (above) using Girsanov's Theorem.

Let \( \tilde{Z}(s) = e^{-\alpha s}Z(s) \) and \( \tilde{X}(s) = e^{-\alpha s}X(s) \) be the process of discounted asset prices and the discounted wealth process, respectively. If the process
\[ \int_0^s \pi(u)d\tilde{Z}(u) \]
is a \( Q \)-local martingale, the discounted process \( (\tilde{X}(s))_{0 \leq s \leq 1} \) is also a \( Q \)-local martingale. We can easily verify that \( \tilde{Z}(s) \) is a \( Q \)-local martingale.

From now on, we will work with the discounted processes \( \tilde{Z}(s) \) and \( \tilde{X}(s) \).

The equivalent martingale measure \( Q \) is not unique in the framework of incomplete markets. There are two widely extended criteria to choose an element of \( M'(P) \) for the calculations: the minimal martingale measure (Definition 11.1) and the variance-optimal measure (Definition 11.2).

**Definition 11.1** Let \( M \in \mathcal{M}^2 \). A martingale measure \( P_{\text{min}} \equiv P \) is called minimal if \( P_{\text{min}} \equiv P \) on \( \mathcal{F}_0 \), and if any square-integrable \( P \)-martingale which is orthogonal to \( M \) under \( P \) remains a martingale under \( P_{\text{min}} \):
\[ L \in \mathcal{M}^2 \text{ and } \langle L, M \rangle = 0 \Rightarrow L \text{ is a martingale under } P_{\text{min}} \]
(16).

We define the variance-optimal martingale measure as follows:

**Definition 11.2** The equivalent measure \( P_c \in M'(P) \) whose density with respect to \( P \) has minimal \( L^2 \)-norm is called the variance-optimal martingale measure ([3]).

12 MANAGEMENT STRATEGY

In Section 7, we developed an actuarial model for a governmental fund for natural disasters. \( \Pi(s) \) is a non-decreasing process representing the aggregated expenditures up to time \( s \in [0,J] \). The total accumulated expenditures at the end of the fiscal year is \( \Pi(J) \).
In order to develop an adequate optimal management strategy to hedge $\Pi(I)$ from the perspective of a risk-averse government, we should outline an adequate optimization problem. We investigated different possibilities and the resulting strategy comes from the solution of Problem 17.1.

The formulation and solution of the main optimization problem (Problem 17.1) is closely related with other problem, which we will call basic optimization problem (Problem 13.1).

13 FORMULATION OF THE BASIC OPTIMIZATION PROBLEM

Define $\tilde{H}(s) = \pi(s) b \tilde{Z}(s), \ s \in [0, I]$, where $\Pi(s)$ is the number of shares to be held at $s$ and $\tilde{Z}(s)$ is the discounted process of prices. Let $\tilde{G}_i(\tilde{H})$ be the cumulative gain process associated to the discounted wealth process $\tilde{X}(s)$.

We have
$$ \tilde{G}_i(\tilde{H}) = \int_0^s d\tilde{X}(u) = \int_0^s \tilde{H}(u)dW(u), \ 0 \leq s \leq 1. \quad (28) $$

Now we define the basic optimization problem.

Problem 13.1: Let $\Theta$ be the space of all investment strategies $\tilde{H}$ such that $\tilde{G}_i(\tilde{H})$ is in the space $\mathcal{S}^2$ of semimartingales. The basic optimization problem is
$$ \min E[(\Pi(I) - c - \tilde{G}_i(\tilde{H}))^2], \ c \in \mathbb{R}, \over line{\text{over all}} \ \tilde{H} \in \Theta. \quad (29) $$

14 SOLUTION OF THE BASIC OPTIMIZATION PROBLEM

We denote the strategy that solves Problem 13.1 as $\tilde{H}^{(c)}$. This strategy has the property to be mean-variance optimal. This type of problems has been often discussed in the literature in various forms of generality in both discrete and continuous settings.

As we mentioned before, from the closeness of $\tilde{G}_i(\Theta)$ in $\mathcal{L}^c$, we know that every r.v. in $\mathcal{L}^c$ has a F-S decomposition. $\Pi(1) \in \mathcal{L}^c$ implies that there is a solution $\tilde{H}^{(c)} \in \Theta$ for all $c$.

The first step is to find the intrinsic value process. We define the intrinsic value process $V_s$ as a square-integrable process with right-continuous paths satisfying $V_0 = E[\Pi(I)]$ and $V_s = \Pi(I) \text{ P-a.s.} \ (see \ [6])$.

The intrinsic value process for our problem is
$$ V_s = E[\Pi(I) | \mathcal{F}_s] = \Pi(s) + E[\Pi(I - s)], \quad (30) $$

where
$$ E[\Pi(s)] = a_1 \tilde{s} + (a_1 - a_2) \sigma \tilde{s} + \tilde{\lambda} s E[\min \{\alpha^{(n)}, \tilde{S}_s\}] + \tilde{\kappa} s E[\min \{\alpha^{(n)}, \tilde{S}_s\}]. \quad (31) $$

We wish to stress that the value of $\Pi(s)$ in Eq. (30) depends only on historical information.
If the minimal martingale measure is also variance-optimal, as it is our case, the solution of Problem 13.1 is given in feedback form as
\[
\tilde{H}^{(c)}_t = \frac{\mu - r}{b} \left[ V_s - c - \int_0^s \tilde{H}^{(c)}_u dW(u) \right].
\]
(32)

Controlling every moment the amount invested in the asset \( \tilde{Z}(s) \) using the solution of Problem 13.1, we influence the path of the portfolio value \( \tilde{X}(s) \). However, it should be noticed that the solution of the basic problem is nonsense from an economic point of view. If the cumulated expenditures fall short, the discounted value of the portfolio will be pulled down. We should not apply the solution of Problem 13.1 in practice. Nevertheless, this problem is useful as a starting point.

Before showing the main optimization problem, let us introduce two closely-related problems.

15 OPTIMAL CHOICE FOR THE INITIAL CAPITAL

Problem 15.1 Let \( \Theta \) be the space of all investment strategies \( \tilde{H} \) such that \( \tilde{G}_j(\tilde{H}) \) is in the space \( S^2 \) of semimartingales. We consider the following optimization problem
\[
\min E[(\Pi(1) - c - \tilde{G}_j(\tilde{H}))^2], \text{ over all } (c, \tilde{H}) \in \Theta.
\]
(33)

[13] showed that the solution of Problem 15.1 is given by \( c^* = \Pi_0 \) and \( \tilde{H}^{c*} \). In our case, the optimal choice for the initial capital in Problem 13.1 is
\[
c^* = E[\Pi(1)].
\]
(34)

16 VARIANCE-MINIMIZING STRATEGY

Problem 16.1: Let \( \Theta \) be the space of all investment strategies \( \tilde{H} \) such that \( \tilde{G}_j(\tilde{H}) \) is in the space \( S^2 \) of semimartingales. We consider the following optimization problem
\[
\min \text{Var}[(\Pi(1) - c - \tilde{G}_j(\tilde{H}))^2], \text{ over all } \tilde{H} \in \Theta.
\]
(35)

This type of problem was solved by [12], [5] and [13] at different levels of generality. The solution of Problem 16.1 is given by the strategy \( \tilde{H}^{c^*} \) (see [13]).

17 MAIN OPTIMIZATION PROBLEM

Outgoing from the solution of Problem 13.1, we worked on the formulation of a related problem that leads to a more realistic investment strategy, feasible from the governmental perspective.

To explain the main idea for the formulation of the problem, let us consider the following citation from [14], p. 35, about reserve funds as governmental ex-ante financing tool:
Reserve funds involve setting aside funds in highly liquid accounts held either domestically or abroad. In theory the annual contribution to that fund should be equal to the annual expected loss of the risk the fund is designed to cover. The cost of these funds is primarily the opportunity cost of not investing the funds elsewhere: highly liquid accounts offer only a 5-6% rate of return compared to the 16% rate of return frequently attributed to investment in development projects.

Our idea for the mathematical formulation is to diminish this opportunity cost through investment in the market. We renounce to absolute high liquidity allocating some resources for investment, but we reduce risk to its intrinsic value using an investment strategy. Our goal is to find an investment strategy that allocates the amount of investment resources in the market strictly necessary to hedge the maximum between the cumulated expenditures \( P(1) \) and the capital resulting from investing \( c^* = E[\Pi(I)] \) in a bank account earning rate \( r \) during the fiscal year.

**Problem 17.1** Let \( \Theta \) be the space of all investment strategies \( \tilde{Q} \) such that \( \tilde{G}_j(\tilde{Q}) \) is in the space \( S \) of semimartingales. The optimization problem is

\[
\min \ E[(\Pi(I) - k - \tilde{G}_j(\tilde{Q}))^2], \ k \in \mathbb{R}, \ \text{over all} \ \tilde{Q} \in \Theta.
\]

where

\[
\Pi^*(I) = \max\{c^* e^r, \Pi(I)\}.
\]

\( c^* e^r \) represents the amount of capital that we would have in the fund if we would invest 100\% of an initial capital \( c^* = E[\Pi(I)] \) in the bank account earning interest rate \( r \). We have chosen the constant \( c^* \) for the formulation of the main problem because it is the optimal starting point to hedge the cumulated expenditures \( \Pi(I) \).

**18 SOLUTION OF THE MAIN OPTIMIZATION PROBLEM**

The solution of Problem 17.1 will be denoted as \( \tilde{Q}^{(k)} \). We solve this problem in an analogous way to Problem 13.1. We should find the intrinsic value process \( V_s^* \) that satisfies \( V_o^* = E[\Pi^*(I)] \) and \( V^*_s = \Pi^*(I) \) \( \text{P-a.s.,} \) where \( \Pi^*(s) = \max\{c^* e^r, \Pi(s)\}, \ s \in [0, I] \).

We define the intrinsic value process \( V_s^* \) as follows:

\[
V_s^* = \Pi^*(s) + E[\Pi^*(I) - \Pi^*(s)].
\]

Equation (38) (above) satisfies the conditions \( V_0 = E[\Pi^*(I)] \) and \( V_I = \Pi^*(I) \).

Finally, the explicit solution of the main optimization problem (Problem 17.1) in feedback form is

\[
\tilde{Q}_s^{(k)} = \frac{\mu - r}{b} \left[ V_s^* - k - \int_0^s \tilde{Q}_u^{(k)} dW(u) \right].
\]
The optimal choice for \( k \) in Problem 17.1 is \( k^* = \mathbb{E}[\Pi^*(I)] \). From \( \mathbb{E}[\Pi^*(I)] > \mathbb{E}[\Pi(I)] \), we know that the optimal starting capital \( k^* \) is higher than \( c^* \). Although Problem 17.1 has a solution for all \( k \in \mathbb{R} \), from an economic perspective, \( c^* \) can be interpreted as the minimal acceptable value of \( k \).

We have conceived Problem 17.1 to be adequate from the public administration point of view. This mathematical model is useful when analyzing quantitatively the feasibility of the established goals at the end of every fiscal year.

![Figure 4: Portfolio value \( X(s) \) resulting from applying the optimal strategy of the basic problem (solid line), optimal amount to be invested \( \tilde{H}^{(\ell)}(s) \) (dashed line) and cumulated expenditures \( \Pi(s) \) (dotted line).](image)

**19 SUBOPTIMAL STRATEGY**

The solution of Problem 17.1 (\( \tilde{Q}^{(k)}(s) \)) has the following disadvantages:

1. It can take negative values.
2. It can demand more resources than we have in our portfolio.

Therefrom, we suggest the use of the following (suboptimal) strategy:

\[
\tilde{I}_s^{(k)} := \min\{\max\{\tilde{Q}_s^{(k)}, 0\}, \tilde{X}^*(s^-)\}.
\]  

(40)

**20 EXAMPLES**

In this section, we show some simulations that will help us visualize the functioning of the strategies solving the basic problem and the main optimization problem. Finally, we give an example of suboptimal strategy and we compare it with the corresponding optimal strategy.
We programmed the solution of the basic problem and of the main optimization problem using Matlab. For the simulations, we retake the settings of Section 8.1. The market parameters are \( r=0.4, \mu=0.15 \) and \( b=0.2 \).

With illustration purposes, all the graphics of this section show the simulated present value processes.

Figure 5: Portfolio value \( X^*(s) \) with initial capital \( k=k^* \) resulting from applying the optimal strategy (solid line), optimal amount to be invested \( \tilde{Q}^{(i)}(s) \) (dashed line), cumulated expenditures \( \Pi(s) \) (dotted line) and \( \Pi^*(s)=\min\{\max\{\Pi(s),0\},\bar{X}^*(s^+)\} \) (dash-dot line).

22.1 EXAMPLE A: THE SOLUTION OF THE BASIC PROBLEM

The solution of Problem 13.1 is valid for all \( c \in \Re \). For the simulation, we used the optimal value for \( c \) \( (c^*=E[\Pi(I)]=816) \) as initial capital to calculate the optimal strategy.

The simulation plotted in Figure 4 evidences the economic nonsense of the basic problem. The optimal amount to be invested at time \( s \) is given by \( H(s)=e^{rs}b \tilde{H}^{(c^+)}(s) \). Observe that the amount to be invested at moment \( s \) is always negative in this example. We can appreciate how the portfolio value \( X(s)=e^{rs} \bar{X}(s) \) is pulled down.

22.2 EXAMPLE B: THE SOLUTION OF THE MAIN OPTIMIZATION PROBLEM

We calculated the expectation of \( \Pi^*(s) \), \( 0 \leq s \leq 1 \), numerically. The estimation of \( E[\Pi^*(s)] \) after 50,000 simulations is 1298.2.

The solution of the main optimization problem (Problem 17.1) is valid for all \( k \in \Re \). For the simulation, we first used the optimal value for \( k \) \( (k^*=E[\Pi^*(I)]=1298.2) \) as initial capital to
calculate the optimal strategy (see Figure 5) and then we tried with $k = c^*$ (see Figure 6), $c^* = 816$.

Figure 6: Portfolio value $X^*(s)$ with initial capital $k = c^*$ resulting from applying the optimal strategy (solid line), optimal amount to be invested $\tilde{Q}^{(k)}(s)$ (dashed line), cumulated expenditures $\Pi(s)$ (dotted line) and $\Pi^*(s) = \min\{\max\{\Pi(s), 0\}, \tilde{X}^*(s^-)\}$ (dash-dot line).

Figure 7: Portfolio value $X^{**}(s)$ with initial capital $k = k^*$ resulting from applying the suboptimal strategy (solid line), suboptimal amount to be invested $I^{(k)}(s)$ (dashed line),
cumulated expenditures $\Pi(s)$ (dotted line) and $\Pi^*(s) = \min\{\max\{\Pi(s), 0\}, X^{**}(s^-)\}$ (dash-dot line).

Figure 8: Comparison of the suboptimal portfolio $X^{**}(s)$ with initial capital $k = k^*$ of Example C (dashed line) vs. the optimal portfolio $X^*(s)$ with initial capital $k = k^*$ (solid line) of Example B.
Figure 9: Portfolio value $X^{**}(s)$ with initial capital $k=c^*$ resulting from applying the suboptimal strategy (solid line), suboptimal amount to be invested $I^{k\dagger}(s)$ (dashed line), cumulated expenditures $\Pi(s)$ (dotted line) and $\Pi^*(s) = \min\{\max\{\Pi(s), 0\}, X^{**}(s^-)\}$ (dash-dot line).

Figure 10: Portfolio value with initial capital $k=k^*$ resulting from applying the optimal strategy (solid line), optimal amount to be invested (dashed line), cumulated expenditures $\Pi(s)$ (dotted line) and $\Pi^*(s) = \min\{\max\{\Pi(s), 0\}, X^{**}(s^-)\}$ (dash-dot line). The optimal and the suboptimal strategies coincide.
Figure 11: Portfolio value with initial capital $k = c^*$ resulting from applying the suboptimal strategy (solid line), suboptimal amount to be invested (dashed line), cumulated expenditures $\Pi(s)$ (dotted line) and $\Pi'(s) = \min\{\max\{\Pi(s), 0\}, X^*(s^-)\}$ (dash-dot line).

22.3 EXAMPLE C: THE SUBOPTIMAL STRATEGY

In this section, we retake the same simulation for the market and the expenditures of Example B and we calculate the suboptimal strategy. Compare Figure 5 vs. Figure 7, and Figure 6 vs. Figure 9. In these examples, the optimal strategy demanded an investment superior to the portfolio value. The suboptimal strategy does not allow an investment higher than the portfolio value.

In Figure 8, we can appreciate the suboptimal and the optimal strategies of Figure 5 (Example B) and Figure 7 (Example C) together.

Finally, we present an example of the case when both strategies coincide (see Figure 10) and a case with low cumulated expenditures and initial capital $k = c^*$.

CONCLUSIONS

The models and the methodologies here presented are the result of an interdisciplinary analysis. They serve as basis for further research in risk management of disasters from an actuarial perspective.

In the first part of this article, we developed a model for the arrival processes of warnings and disasters exploiting information about the performance of an early warning system. Although the assumption of independent claims and expenditures is too simple to be realistic, this kind of models (e.g. the Lundberg model) have proven its usefulness to develop techniques for more general risk processes.

The decomposition of the involved arrival processes into independent processes has a variety of applications. Among them, the quantification of the economic impact of natural disasters in a governmental fund for disasters.

The implementation of a natural disasters fund by itself is not enough to reduce substantially the economic vulnerability of a risk-averse country if we are not able to take advantage of it in the long-term planning. In the last Section, we developed an optimal management strategy for the wealth $X(s)$ in order to hedge the total accumulated expenditure at the end of the fiscal year $\Pi(t)$. We showed that investment combined with XL-reinsurance is a plausible alternative for risk-averse governments to improve the risk management of natural disasters funds and, therefore, improve long-term planning. Our results evidence that building reserves is justified in the framework of a risk-averse government. The reserves are important because they enable a better budget planning and they reduce economic vulnerability in the long-term. A topic for further research is to solve the optimization problem at different levels of generalization and to incorporate other alternatives for risk management in the modelling. We
can also calculate optimal strategies for the reinsurance company and investigate the interaction of the governmental strategy with the strategy of a reinsurance company.

In some countries, the budget for social programs is shortened in order to cover catastrophic losses. A better budget planning and reserve management using ad hoc actuarial technics can enable the possibility to stabilize also the budget for diverse social programs, like poverty mitigation.

In the case of catastrophic events, government and private capitals often share risks. Nevertheless, it is till now unclear, how this risk-partnership should be characterized. Consequently, it is essential to develop risk management methodologies from a governmental perspective. Only after this first step, we would be able to develop risk-partnerships models for the absorption of catastrophic losses. The well-understanding of the interaction between insurance industry, financial markets and government can mean the line between insurability and non-insurability of catastrophic losses.

The incorporation of the notion of early warning systems in actuarial models as well as the development of methodologies for governmental risk management are two topics that, in our opinion, are of fundamental importance for the actuarial profession.

BIBLIOGRAPHY


