LEVEL OF PRUDENCE IN CLAIMS PROVISIONS AND CAPITAL ADEQUACY OF NON-LIFE INSURANCE COMPANIES

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Abstract
A business line of a non-life insurance company is considered. A model of the evolution of its loss reserves under a prudential reserving rule is presented. An equation is derived for the amount of allocated free capital needed to cover the underwriting and the investment risks with a given confidence level. Other types of risk are, for the sake of simplicity, assumed to be taken into consideration by choosing properly the parameters of the two basic risks.

Key words
prudential reserving rules, risk based capital, Solvency II, internal risk models

At the time when this paper was written the most up to date information regarding the relevant topics of the Solvency II project was obtainable from [1], where the prudent reserving and the confidence level for the solvency capital requirement are explained and analyzed.

We shall deal with a block of business of a non-life insurance company. In the first part of the paper we present a model of the evolution of the loss reserves under a prudential reserving rule. In the second part we give an equation to determine the allocated free capital needed to attain a required confidence level. We are of the opinion stated in [6] that a considerable defect of the existing risk based capital formulas is the absence of an underlying probabilistic model.

We follow the principle of parsimony in mathematical modelling (see [3], par. 5.1). A parsimonious model makes it possible to concentrate on the estimation of its parameters, which is more meaningfull than to combine complex estimates in a manner violating the laws of probability theory.

For the sake of brevity only the underwriting risk and the investment risk are considered. It is assumed that the credit risk, the operational risk etc. are taken into account when choosing the parameters of the two basic risks.

We believe that the paper contributes to a rather neglected component of the Solvency II project, namely to the development of actuarial approaches applicable to internal risk models.

Evolution of the loss reserve

Let the block of business in consideration be stable enough to make it possible to use the development triangles to obtain the loss development factors together with an estimate of the variance of the loss reserves. Introduce the triangle of *cummulative paid losses net of reinsurance* at the end of year (period) *t*

$$(\Delta) \qquad \begin{array}{llll} X_{t-r,0} & X_{t-r,1} & \cdots & X_{t-r,r} \\ \vdots & \vdots & & \\ X_{t-1,0} & X_{t-1,1} & & \\ X_{t,0} & & & \end{array}$$

We assume $X_{t-r,r} = X_{t-r,\infty}$, at the end of development year *r* the loss payments are completed. The rows of the triangles, which contain the development of losses pertaining to different years of origin, are suposed to be mutually independent. Let development factors $c_0, c_1, ..., c_{r-1}$ be given such that

$$\mathrm{E}\{X_{s,j+1} \mid X_{s,0}, ..., X_{s,j}\} = X_{s,j} c_j \qquad (1)$$

holds in accordance with the *chain ladder* method.

To define the deviations from the expected values we use the following multiplicative model

$$X_{s,j+1} = X_{s,j}(1+(c_j-1)E^{j+1}_{s+j+1}) \ .$$

Random variables $\{E^j_{s+j}\}$ are supposed to be independent mutually and independent of $\{X_{s,0}\}$. (1) implies $\mathrm{E}E^j_{s+j} = 1$. Further let $\{E^j_{s+j}\}$ for fixed *j* have same first three moments. We have

$$X_{s,1} = X_{s,0}(1+(c_0-1)E^1_{s+1}), \dots , X_{s,j} = X_{s,0}(1+(c_0-1)E^1_{s+1})\dots(1+(c_{j-1}-1)E^j_{s+j}),$$

and

$$\mathrm{E}\{X_{t-j,\infty} \mid \Delta\} = X_{t-j,j}c_j c_{j+1}\dots c_{r-1} \ .$$

The expected value of future loss payments at time *t* equals

(2) $$\sum_{j=0}^{r-1} X_{t-j,j}(c_j\dots c_{r-1}-1),$$

which is the proper value the *undiscounted loss reserve*.

Our aim is to consider additional *prudence in the loss reserves*. Prudence is mostly interpreted as a postulate to augment the undiscounted reserve by a multiple of the standard deviation of the remaining amount to be paid for losses. Hence, the total reserve equals

(3) $$\sum_j \mathrm{E}\{X_{t-j,\infty} - X_{t-j,j} \mid \Delta\} + z_\alpha \sqrt{Var(\sum_j (X_{t-j,\infty} - X_{t-j,j}) \mid \Delta)} \ ,$$

where α denotes the *level of prudence* and z_α is a quantile of the standard normal distribution, e. g., $z_{0.75} = 0.6745$. Standard deviation is a commonly accepted measure of risk. z_α is a parameter whose scaling by α is implied by the role of the normal distribution in probability theory.

Let us quote from [2], par. 15 : „A quantitative benchmark for the confidence level of technical provisions should be set at 75% of the probability distribution of the claims." With regard to the data on the loss development available normally for statistical inference, (3) is a reasonable practical way how to implement this requirement.

The first term in (3) is given by (2). To determine

$$Var(X_{t-j,\infty} \mid \Delta) = Var(X_{t-j,r} - X_{t-j,j} \mid \Delta)$$

we define

$$c_k^{(m)} = \mathrm{E}(1+(c_k-1)E^{k+1}_{s+k+1})^m, \quad m = 2, 3,$$

and obtain

(4) $$Var(X_{t-j,\infty} \mid \Delta) = \mathrm{E}\{X^2_{t-j,r} \mid \Delta\} - \mathrm{E}\{X_{t-j,r} \mid \Delta\}^2 = X^2_{t-j,j}v_j \ ,$$

where $$v_j = c_j^{(2)}\dots c_{r-1}^{(2)} - c_j^2\dots c_{r-1}^2 \ .$$

Inserting into (3) from (2), (4), and with regard to the independence of the rows in the development traingle the loss reserve at the end of year *t* obtains in the form

(5) $$\sum_j X_{t-j,j}(c_j ... c_{r-1} - 1) + z_\alpha \sqrt{\sum_j X^2_{t-j,j} v_j} \ .$$

Mathematical treatment of the second term in (5) is complicated. We therefore replace it by the first order expansion at the point

$$(\mathrm{E}X_{t,0}, \mathrm{E}X_{t-1,1}, ..., \mathrm{E}X_{t-r+1,r-1}) .$$

The modeled amount of the loss reserve at the end of year t corresponding to level of prudence α is then

(6) $$R_t = \sum_j X_{t-j,j} W_t^j \text{ , where } W_t^j = c_j ... c_{r-1} - 1 + z_\alpha \frac{\mathrm{E}X_{t-j,j} v_j}{\sqrt{\sum_j (\mathrm{E}X_{t-j,j})^2 v_j}} \ .$$

The additional reserve established with regard to the prudence equals

(7) $$P_t = z_\alpha \sum_j X_{t-j,j} \frac{\mathrm{E}X_{t-j,j} v_j}{\sqrt{\sum_j (\mathrm{E}X_{t-j,j})^2 v_j}} .$$

For the data on the diagonal of the development triangle at time t we introduce the denotation

(8) $$X_{t,0} = {}^0X_t, \ldots, X_{t-j,j} = {}^jX_t, \ldots, X_{t-r+1,r-1} = {}^{r-1}X_t \ .$$

Since the random variables $\{E_t^j, {}^0X_t\}$ are mutually independent, it holds that given the values of the variables (8), the future loss development is independent of the development until time t. In this sense the vector $({}^0X_t, ..., {}^{r-1}X_t)$ represents *the state* of the modelled system at time t.

The centered random variables will be denoted by a stripe, e. g., ${}^j\overline{X}_t = {}^jX_t - \mathrm{E}\,{}^jX_t$.

Only the values of the first three moments of random variables will be needed in the formulas derived in this paper. From

(9) $${}^jX_{t+1} = {}^{j-1}X_t (1 + (c_{j-1} - 1) E_{t+1}^j)$$

expressing the second and third moments by the central ones and vice versa one obtains, for $j = 1, ..., r-1$,

(10) $$\begin{aligned}
\mathrm{E}\,{}^jX_{t+1} &= \mathrm{E}\,{}^{j-1}X_t c_{j-1} \ , \\
\mathrm{E}\,{}^j\overline{X}_{t+1}^2 &= \mathrm{E}\,{}^{j-1}\overline{X}_t^2 c_{j-1}^{(2)} + (\mathrm{E}\,{}^{j-1}X_t)^2 (c_{j-1}^{(2)} - c_{j-1}^2) , \\
\mathrm{E}\,{}^j\overline{X}_{t+1}^3 &= \mathrm{E}\,{}^{j-1}\overline{X}_t^3 c_{j-1}^{(3)} + \mathrm{E}\,{}^{j-1}\overline{X}_t^2 \mathrm{E}\,{}^{j-1}X_t (3c_{j-1}^{(3)} - 3c_{j-1}^{(2)} c_{j-1}) + \\
&\quad + (\mathrm{E}\,{}^{j-1}X_t)^3 (c_{j-1}^{(3)} - 3c_{j-1}^{(2)} c_{j-1} + 2c_{j-1}^3) , \\
c_{j-1}^{(2)} &= \mathrm{E}(1 + (c_{j-1} - 1)E^j)^2 \ , \ c_{j-1}^{(3)} = \mathrm{E}(1 + (c_{j-1} - 1)E^j)^3 .
\end{aligned}$$

To model the evolution of the loss reserves during N periods one has to specify the initial state $({}^0x_0, {}^1x_0, \ldots, {}^{r-1}x_0)$ and the inputs $(\mathrm{E}^0X_t, \mathrm{E}^0\overline{X}_t^2, \mathrm{E}^0\overline{X}_t^3)$, $t = 1, \ldots, N$.

The risk reserve

According to [2], par. 16 the solvency capital requirement should be based on the amount of economic capital corresponding to a ruin probability $\varepsilon = 0.5\%$ and a one year time horizon. Ruin occurs when the amount of admissible assets is lower than the amount of technical provisions. In [6] we have pointed out that even in much simpler situations it is not possible to define tail events of such a small probability with reasonable accuracy and that ε should be regarded as a parameter whose value should be determined aposteriori from a selected class of insurance companies. In [1] par. 2.14 it is stated that $\varepsilon = 0.5\%$ may be broadly equivalent to an investment grade rating of the company.

As in the preceding section we consider a block of business. We outline a procedure to determine the amount of allocated surplus needed to achieve the desired level ε of the ruin probability. The time horizon is one year. We deal therefore with a *one period model*. The following denotations will be used

u_0 – allocated surplus at the inception of the period;

U_1 – the risk reserve at the end of the period;

I – net rate of return on the financial placement of allocated capital, premiums and technical provisions. The expected value, standard deviation and skewness of I should reflect the composition of the investment portfolio. I is assumed to be independent of $\{{}^0X_1, E_{s+j}^j\}$.

$F = 1 + I$;

γ – average date of the loss payments in the year; for uniformly distributed payments $\gamma = 0.5$;

$G = 1 + (1-\gamma)I$;

B_1 – the earned risk premiums (the earned premiums after subtraction of the calculated operation expenses). B_1 is assumed to be a nonrandom quantity, decomposed into the earned premiums from previous years and earned new premiums, $B_1 = B_1^0 + B_1^1$. If the investment return is added, the value of the earned premiums at the end of the period obtains as

(11) $$B_1^0(1+I) + B_1^1(1+(1-\beta)I),$$

where β is the average time of the earned premium payment;

$H = 1 + (1-\beta)I$.

It is assumed that the operation expenses are incurred as calculated in the premiums. *The risk reserve at the end of the period* is then the sum of the allocated capital, the earned risk premiums, the difference of the initial a the final loss reserves diminished by the paid losses and adjusted with regard to the investment earnings. Thus we can write

$$U_1 = u_0F + B_1^0F + B_1^1H + \sum_j {}^jx_0W_0^jF - \sum_j {}^jX_1W_1^j - [{}^0X_1 + \sum_{j=1}^{r-1}({}^jX_1 - {}^{j-1}x_0) + {}^{r-1}x_0E_1^r(c_{r-1}-1)]G.$$

The last term in square brackets represents paid losses from year –*r*+1, the before last term is the sum of paid losses from years 0,…, –*r*+2 and then there are the paid losses from year 1. From the last equation we obtain

(12) $$\mathrm{E}U_1 = (u_0 + B_0^1 + \sum_j {}^j x_0 W_0^j)\mathrm{E}F + B_1^1 \mathrm{E}H - \sum_j \mathrm{E}\,{}^j X_1 W_1^j - (\mathrm{E}\,{}^0 X_1 + \sum_j {}^j x_0 (c_j - 1))\mathrm{E}G .$$

Denote

$$A = u_0 + B_0^1 + B_1^1(1-\beta) + \sum_j {}^j x_0 W_0^j - (\mathrm{E}\,{}^0 X + \sum_j (c_j - 1)\,{}^j x_0)(1-\gamma),$$

$${}^r X_1 = {}^{r-1}x_0 E_1^r (c_{r-1} - 1).$$

The centered variable $\overline{U}_1$ can be expressed as follows

(13) $$\overline{U}_1 = A\bar{I} - \sum_j {}^j \overline{X}_1 (W_1^j + \mathrm{E}G) - {}^r\overline{X}_1 \mathrm{E}G - (\sum_j {}^j \overline{X}_1 + {}^r\overline{X}_1)(1-\gamma)\bar{I} .$$

Using the assumption that $\{{}^j\overline{X}_1, \overline{E}_1^r, \bar{I}\}$ are mutually independent with zero expectation one obtains from (13) the following formulas for moments

(14) $$\mathrm{E}\overline{U}_1^2 = A^2\mathrm{E}\bar{I}^2 + \sum_j (W_1^j + \mathrm{E}G)^2 \mathrm{E}\,{}^j\overline{X}_1^2 + (\mathrm{E}G)^2 \mathrm{E}\,{}^r\overline{X}_1^2 + (\sum_j \mathrm{E}\,{}^j\overline{X}_1^2 + \mathrm{E}\,{}^r\overline{X}_1^2)(1-\gamma)^2 \mathrm{E}\bar{I}^2 ,$$

(15) $$\begin{aligned}\mathrm{E}\overline{U}_1^3 &= A^3\mathrm{E}\bar{I}^3 - \sum_j (W_1^j + \mathrm{E}G)^3 \mathrm{E}\,{}^j\overline{X}_1^3 + (\mathrm{E}G)^3 \mathrm{E}\,{}^r\overline{X}_1^3 - (\sum_j \mathrm{E}\,{}^j\overline{X}_1^3 + \mathrm{E}\,{}^r\overline{X}_1^3)(1-\gamma)^3 \mathrm{E}\bar{I}^3 - \\ &- 3(\sum_j (W_1^j + \mathrm{E}G)\mathrm{E}\,{}^j\overline{X}_1^3 + \mathrm{E}G\mathrm{E}\,{}^r\overline{X}_1^3)(1-\gamma)^2 \mathrm{E}\bar{I}^2 + \\ &+ 6(\sum_j (W_1^j + \mathrm{E}G)\mathrm{E}\,{}^j\overline{X}_1^2 + \mathrm{E}G\mathrm{E}\,{}^r\overline{X}_1^2) A(1-\gamma)\mathrm{E}\bar{I}^2 ,\end{aligned}$$

where with regard to (10)

$\mathrm{E}\,{}^j X_1 = {}^{j-1}x_0 c_{j-1}$, $\mathrm{E}\,{}^j\overline{X}_1^2 = {}^{j-1}x_0^2 (c_{j-1}^{(2)} - c_{j-1}^2)$, $\mathrm{E}\,{}^j\overline{X}_1^3 = {}^{j-1}x_0^3 (c_{j-1}^{(3)} - 3c_{j-1}^{(2)} c_{j-1} + 2c_{j-1}^3)$, $j = 1,\ldots,r\text{-}1$.

The conditon defining *the solvency capital requirement* u_0 is

(16) $$P(U_1 < 0) = \varepsilon .$$

Expressing the ε-quantile of the distribution of U_1 by means of the NP2 approximation ([4]), we get from (16)

(17) $$0 = \mathrm{E}U_1 - z_{1-\varepsilon}\sqrt{\mathrm{E}\overline{U}_1^2} + \frac{\mathrm{E}\overline{U}_1^3}{6\mathrm{E}\overline{U}_1^2}(z_{1-\varepsilon}^2 - 1) .$$

Substituting for the moments in (17) according to (12), (14) and (15) we determine u_0.

Omitting the last term in (17) one obtains the quantile of U_1 under the approximation by the normal distribution (NP1 approximation)

$$0 = \mathrm{E}U_1 - z_{1-\varepsilon}\sqrt{\mathrm{E}\overline{U}_1^2} .$$

It is questionable whether the identification of the event $\{U_1 < 0\}$ with a ruin is correct, since the company has built the additional reserve P_1. In our opinion a ruin would be more adequately represented by the event $\{U_1 + P_1 < 0\}$ and by u_0 determined from

$$P(U_1 + P_1 < 0) = \varepsilon \ .$$

The calculation of u_0 in this case is analogous to the procedure described above. One has only to replace z_α in (6) for $t = 1$ by zero.

Example. To illustrate the explained methods we shall use the development triangle pertaining to a UK Motor Non-Comprehensive account published in [5]. It contains the following amounts of cummulative paid losses

	0	1	2	3	4	5	6
-6	3511	6726	8992	10704	11763	12350	12690
-5	4001	7703	9981	11161	12117	12746	
-4	4355	8287	10233	11755	12993		
-3	4295	7750	9773	11093			
-2	4150	7897	10217				
-1	5102	9650					
0	6283						

For $\alpha = 0.75$, $\varepsilon = 0.005$ we shall determine the amount of the allocated surplus u_0 needed, and shall consider the influence of the definitions of ruin, $\{U_1 < 0\}$ or $\{U_1 + P_1 < 0\}$, and of the NP2 or NP1 approximations of quantiles on the resulting u_0.

By means of chain ladder method the development factors are estimated as follows

$$c_0 = 1.8892 \ , \ c_1 = 1.2824 \ , \ c_2 = 1.1471 \ , \ c_3 = 1.0968 \ , \ c_4 = 1.0509 \ , \ c_5 = 1.0275 \ , \ c_6 = 1 \ ,\ldots$$

Taking the values obtained for parameters of the model, we get from (9)

$$E_{s+j+1}^{j+1} = \frac{X_{s,j+1} / X_{s,j} - 1}{c_j - 1} \ .$$

Assume that the probability distribution of E_{s+j+1}^{j+1} does not depend on j. From the development triangle one computes that $\frac{1}{21}\sum E_{s+j+1}^{j+1}$ = 1.0032, which is in good agreement with the assumption $\mathrm{E}E_{s+j+1}^{j+1} = 1$, and further

$$\tfrac{1}{21}\sum (E_{s+j+1}^{j+1} - 1)^2 = 0.01208, \qquad \tfrac{1}{21}\sum (E_{s+j+1}^{j+1} - 1)^3 = 0.000897.$$

The first column of the triangle contains the observed values of the input variables $X_{s,0}$. The values indicate that it is plausible to consider the regression model

$$X_{s,0} = a + by_s + \eta_s \ , \ \text{where} \ y_s = 0, s = -6, \ldots ,-2, \quad y_{-1} = 1, \quad y_0 = 2 \ ,$$

and $\{\eta_s\}$ are deviations from regression. Regression analysis produces a = 4057, b = 1099 and residual variance 91 810.

Let us suppose that the increase in loss volume in the years -1, 0 was caused by the increase of earned premiums, which is expected to persist in year 1. The above analysis implies the choice $\mathrm{E}X_{1,0} = a + 3b = 7335$, $VarX_{1,0} = 91\,810$. Assuming that $X_{1,0}$ has lognormal distribution, we have

$$X_{1,0} = e^{\xi}, \xi \approx N(8.9023\ , 0.0017).$$

Further model parameters were chosen as follows

$$B_1^0 = 11400, \quad B_1^1 = 12800, \quad \mathrm{E}(\overline{E}_{s+j+1}^{j+1})^2 = 0.01208, \quad \mathrm{E}(\overline{E}_{s+j+1}^{j+1})^3 = 0.000897,$$
$$\mathrm{E}I = 0.4, \quad \mathrm{E}\overline{I}^2 = 0.0001, \quad \mathrm{E}\overline{I}^3 = 0.000001, \quad \beta = 0.5, \quad \gamma = 0.5\,.$$

The values of u_0 under the two definitions of ruin and for the two kinds of approximation are presented in the following array

	$\{U_1 < 0\}$	$\{U_1 + P_1 < 0\}$
NP2	3185	2054
NP1	2776	1647 ,

$\mathrm{E}R_1 = 34699, \quad \mathrm{E}P_1 = 1093$.

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