Largest claims reinsurance: the cedant's point of view

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Abstract: We present results allowing one to evaluate the cost and the variance reduction of the cedant in the framework of reinsurance treaties based on order statistics. We compare the efficiency of such treaties with excess of loss covers. Numerical examples show that the best choice may depend on the distribution of the individual claim size, especially on the heaviness of the tail. It may also depend on the premium principles applied by both companies.

Keywords: Largest claims reinsurance, excess of loss reinsurance, optimal reinsurance, order statistics

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1. Introduction

A lot of papers have been devoted to the pricing of reinsurance treaties based on the largest claims during a given time period. Especially, since the early eighties E. Kremer have dealt with this problem in a series of papers in various situations, using several methods. More recently, Berglund [Berg] and Walhin [Wa] have also contributed to this subject. Essentially, the goal of these authors was the pricing of LCR and/or the ECOMOR treaties. In particular, calculations or estimations of the first two moments of the reinsurer claim expenses were developed.

In the present paper, our aim is to examine the impact of LCR and ECOMOR reinsurance treaties from the point of view of the reinsured, i.e. the cedant. We provide formulas allowing one to calculate the pure premium and the standard deviation of the reinsured. Numerical examples are also provided. They show that Largest Claims Reinsurance and ECOMOR covers are generally not better than XL Reinsurance treaties when the latter have an infinite upper limit. Only in some cases LCR treaties are a little more efficient than XL ones. The heaviness of the tail of the claim size distribution is seen to play an important part. Two types of premium principles are considered in this context, especially the expectation principle and the standard deviation principle. On the other hand, the number of largest claims ceded to the reinsurer is also relevant.

In Section 2, we present the needed definitions and preliminaries. In Section 3 we establish general formulas for the first two moments of the cedant's share for LCR and ECOMOR treaties. In Section 4, we compare LCR and ECOMOR treaties with an excess of loss (XL) treaty, under two possible choices of the premium principle. Section 5 contains numerical examples. Two different claim size distributions are considered: the translated exponential and the generalized Pareto. Our general conclusions are presented in Section 6. As to the references list, in addition to papers related to our work, we have mentioned some recent books dealing with non life insurance or reinsurance mathematics.

2. Notations, Definitions, and Preliminaries

For a given class of risks and a given period of time, N denotes the number of claims and C_1 , C_2 , ..., C_n , ... denotes the claims sizes. The random variables C_n are assumed to be nonnegative, independent and identically distributed, and the number of claims is assumed to be independent from the sequence of the claim sizes. All the random variables are assumed to be defined on a probability space (Ω, A, P). For each integer $n \geq 0$, we set

$$p_n = P(N = n).$$

Further, we denote by F the common distribution function of the claims sizes and we assume the existence of a density f. For every integer $n \geq 1$, the sequence

$$C_{1:n} \leq C_{2:n} \leq \ldots \leq C_{n:n}$$

stands for the sequence of the first n claims sizes in the increasing order. In particular

$$C_{1:n} = \min(C_1, C_2, ..., C_n) \quad \text{and} \quad C_{n:n} = \max(C_1, C_2, ..., C_n)$$

are respectively the smallest and the largest claim size among $C_1, C_2, ..., C_n$. We also set $C_{i:n} = 0$ if $i < 0$ or $i > n$. For each integer k such that $1 \leq k \leq n$, $F_{k:n}$ denotes the cumulative distribution function of $C_{k:n}$.

In the framework of the collective risk model, the total aggregate claim amount, denoted by X, has the following form

$$X = \sum_{i=1}^{N} C_i \quad \text{if } N > 0, 0 \quad \text{if } N = 0.$$

As is well known, the first two moments of X are given by

(2.1) $$E(X) = E(N)\, E(C)$$

and

(2.2) $$\mathrm{Var}(X) = E(N)\, \mathrm{Var}(C) + \mathrm{Var}(N)\, E(C)^2.$$

In particular, when the distribution of N is Poisson with parameter λ, the above formulas reduce to

(2.3) $$E(X) = \lambda\, E(C) \quad \text{and} \quad \mathrm{Var}(X) = \lambda\, E(C^2).$$

Assume that a reinsurance treaty has been concluded between a reinsurer and a reinsured (a cedant). We denote by X' (resp. X") the total aggregate claim amount paid by the reinsured (resp. the reinsurer). These random variables obviously satisfy

$$X = X' + X''.$$

Basically, a reinsurance treaty based on ordered claim sizes is given by a sequence of functions R_n of the following type

(2.4) $$R_n(c_1, ..., c_n) = \sum_{i=1}^{n} h_i(c_{i:n}) \qquad n \geq 1$$

where $(h_i)_{i\geq 1}$ denotes a given sequence of measurable functions

$$h_i : [0, +\infty) \rightarrow [0, +\infty)$$

verifying

$$0 \leq h_i(x) \leq x \qquad x \geq 0.$$

In such a case, X' and X" read as follows

(2.5) $$X'' = R_N(C_1, ..., C_N) = \sum_{i=1}^{N} h_i(c_{i:N}) \qquad \text{and} \quad X' = X - X''.$$

More precisely, X" takes on the value $R_n(c_1, ..., c_n)$ if N = n and $C_i = c_i$ for i = 1, ..., n. Let us mention the following three examples, where p denotes a positive integer. For convenience, it is assumed that $p \leq N$.

(i) The largest claims reinsurance treaty of order p, denoted by LCR(p), is defined by

$$X'' = \sum_{j=1}^{p} C_{N-j+1:N}$$

Here, formula (2.4) holds with

$$h_i(x) = x \qquad \text{if } i = N-p+1, ..., N$$

$$h_i(x) = 0 \qquad \text{if } i \leq N-p$$

so that the reinsurer will pay the p largest claims that have occurred during a given period of time.

(ii) The ECOMOR(p) treaty is defined by

$$X'' = \sum_{j=1}^{p} \left(C_{N-j+1:N} - C_{N-p+1:N}\right) \qquad = \sum_{j=1}^{p-1} C_{N-j+1:N} - (p-1)\, C_{N-p+1:N}$$

for $p \geq 2$. For p = 1, we have X" = 0. In this case, formula (2.5) is valid with

$$h_i(x) = x \qquad \text{if } i = N-p+2, ..., N$$

$h_i(x) = (1-p)\, x \qquad \text{if } i = N-p+1$

$h_i(x) = 0 \qquad \text{if } i \leq N-p$

Clearly, the p-th largest claim size plays the role of a random priority (or deductible).

(iii) The Drop Down Excess of Loss treaty is a variant of an excess of loss (XL) treaty, where the layer may vary in function of the order of the claim, after the claims have been ordered by increasing order. The reinsurer's share has the following form

$$X'' = \sum_{i=1}^{N} h_i(C_{N-i+1:N}) \quad .$$

For example, assume that two different excess of loss covers are involved and that some integer p is given ($1 \leq p \leq N$). Then, formula (2.5) still holds with

$$h_i(x) = \min(t_1 - s_1 \,,\, \max(0, x - s_1)) \qquad \text{if } i = 1, ..., p-1$$

and

$$h_i(x) = \min(t_2 - s_2 \,,\, \max(0, x - s_2)) \qquad \text{if } i = p, ..., N$$

where s_1 (resp. s_2) and t_1 (resp. t_2) denote the priority and the upper limit of the first (resp. second) XL treaty.

3. The first two moments of the reinsured share in the LCR and ECOMOR treaties

Since the total claims amount X satisfies $X = X' + X''$ where X' (resp. X'') is the reinsured (resp. reinsurer) share, we immediatly get

(3.1) $$E(X) = E(X') + E(X'')$$

which allows us to deduce the expectation of the reinsured share once the expectation of the reinsurer share is known. In the LCR(p) treaty one has

(3.2) $$E(X'') = \sum_{j=1}^{p} E(C_{N-j+1:N})$$

and in the ECOMOR(p) treaty

(3.3) $$E(X'') = \sum_{j=1}^{p} E(C_{N-j+1:N}) - p\, E(C_{N-p+1:N}) \quad .$$

As to the variance of X', we have in both cases

(3.4) $$\mathrm{Var}(X') = \mathrm{Var}(X - X'') = \mathrm{Var}(X) + \mathrm{Var}(X'') - 2\,\mathrm{cov}(X, X'')$$

where

(3.5) $$\mathrm{cov}(X, X'') = E(X\,X'') - E(X)\,E(X'').$$

Formulas for the expectation E(X") and variance Var(X") have been proposed in several works, especially in that of Berglund [Ber]. Thus, in view of (3.4) and (3.5) it only remains to calculate E(X X").

Berglund's formulas are recalled hereafter, because they are used in our numerical examples in Section 5 and because the formula for E(X X") is more or less of the same type. The k-th moment of $C_{i:N}$ is given by

$$E\big((C_{i:N})^k\big) = \frac{1}{(i-1)!} \int_0^1 F^{-1}(u)^k\,(1-u)^{i-1}\,\psi^{(i)}(u)\,du$$

where $1 \leq i \leq n$ and $k \geq 1$. As to the expectation of the cross product of the i-th and the j-th ordered claim sizes, satisfying $1 \leq i < j \leq n$ we have

$$E(C_{i:N}\,C_{j:N}) =$$

$$\frac{1}{(i-1)!\,(j-i-1)!} \int_0^1 F^{-1}(v)\,(1-v)^{j-1}\,\psi^{(j)}(v) \int_0^1 F^{-1}\big(1-u(1-v)\big)\,u^{i-1}\,(1-u)^{j-i-1}\,du\,dv \quad .$$

Before stating the first result of this section, it is useful to define the expression denoted by $E_N(C_1\,C_{N-j+1})$ for each j = 1, ..., p. It is the random variable taking on the value $E(C_1\,C_{n-j+1})$ with probability $p_n = P(N = n)$.

Theorem 3.1

(a) In the LCR(p) treaty the expectation of X X" is given by

$$E(X\,X'') = \sum_{j=1}^{p} E(N\,E_N(C_{N-j+1:N}\,C_1)) \quad .$$

(b) In the ECOMOR(p) treaty, we have

$$E(X\,X'') = \sum_{j=1}^{p-1} E\big(N\,E_N(C_1\,C_{N-j+1:N})\big) - (p-1)\,E\big(N\,E_N(C_1\,C_{N-p+1:N})\big) \quad .$$

Proof. In the LCR(p) treaty we have

(3.6) $$E(X X'') = E\Big\{\Big(\sum_{i=1}^{N} C_i\Big)\Big(\sum_{j=1}^{p} C_{N-j+1:N}\Big)\Big\} = \sum_{j=1}^{p} E\Big(C_{N-j+1:N} \sum_{i=1}^{N} C_i\Big).$$

For each integer $j = 1, ..., p$ and each $n \geq 1$, the hypotheses of the collective risk model allow us to write

$$E\Big(C_{N-j+1:N} \sum_{i=1}^{N} C_i \;/\; N = n\Big) = E\Big(C_{n-j+1:n} \sum_{i=1}^{n} C_i\Big) = \sum_{i=1}^{n} E(C_{n-j+1:n}\, C_i).$$

The important point is that, due to the exchangeability of the sequence $(C_i)_{i\geq 1}$ we have

$$E\Big(C_{N-j+1:N} \sum_{i=1}^{N} C_i \;/\; N = n\Big) = n\, E(C_{n-j+1:n}\, C_1)$$

which entails

$$E\Big(C_{N-j+1:N} \sum_{i=1}^{N} C_i \;/\; N\Big) = N\, E_N(C_{N-j+1:N}\, C_1)$$

where, for each $j = 1, ..., p$, $E_N(C_1\, C_{N-j+1})$ is defined as above. It follows

$$E\Big(C_{N-j+1:N} \sum_{i=1}^{N} C_i\Big) = E\Big\{E\Big(C_{N-j+1:N} \sum_{i=1}^{N} C_i \;/\; N\Big)\Big\}$$

$$= E\Big(N\, E_N(C_{N-j+1:N}\, C_1)\Big)$$

which in view of (3.6) yields the desired result. The formula for the ECOMOR(p) treaty is derived similarly. Q.E.D.

Our task is now to calculate the expectations involved in formulas (a) and (b) in Theorem 3.1. Observe first that for each $j = 1, ..., p$

(3.7) $$E(N\, E_N(C_{N-j+1:N}\, C_1)) = \sum_{n\geq j} n\, p_n\, E(C_1\, C_{n-j+1:n}) \quad .$$

In order to calculate

$$I_{k:n} := E(C_1\, C_{n-j+1:n}),$$

we need to know the distribution of the pair $(C_1 , C_{k:n})$, where $k = n-j+1$ ranges from 1 to n. At this point, it is important to observe that the distribution $\mu_{k:n}$ of the pair $(C_1 , C_{k:n})$ is not absolutely continuous with respect to the Lebesgue measure λ_2 on the measurable space $(\mathbf{R}^2, B(\mathbf{R}^2))$, because the event $\{C_1 = C_{k:n}\}$

has a non zero probability (namely 1/n). The distribution $\mu_{k:n}$ has the following form

$$(3.8) \qquad \mu_{k:n} = \mathbb{1}_{\Delta}(x, y)\, a(x, y)\, \lambda_2 + \mathbb{1}_{\Delta'}(x, y)\, b(x, y)\, \lambda_2 + \mathbb{1}_{D}(x)\, c(x)\, \lambda_D$$

where λ_2 denotes the Lebesgue measure on $(\mathbf{R}^2, B(\mathbf{R}^2))$ and λ_D the Lebesgue measure on the line $D = \{(x, y) \text{ in } \mathbf{R}^2 : x = y\}$. The subsets Δ and Δ' are defined by

$$\Delta = \{(x, y) \text{ in } \mathbf{R}^2 : x < y\} \qquad \text{and} \qquad \Delta' = \{(x, y) \text{ in } \mathbf{R}^2 : x > y\}.$$

On Δ, namely when $x < y$, standard differential and combinatorial arguments (see e.g. [DN] p. 11) allow us to derive the formulas

$$a(x, y) = \frac{(n-1)!}{(k-2)!\,(n-k)!} \quad f(x)\, f(y)\, F(y)^{k-2}\, (1-F(y))^{n-k}$$

valid for k = 2, 3, ..., n. On Δ', namely when $x > y$, we get similarly

$$b(x, y) = \frac{(n-1)!}{(k-1)!\,(n-k-1)!} \quad f(x)\, f(y)\, F(y)^{k-1}\, (1-F(y))^{n-k-1}$$

valid for k = 1, 2, ..., n-1. On the line D, function c(.) given by

$$c(x) = \frac{(n-1)!}{(k-1)!\,(n-k)!} \quad f(x)\, F(x)^{k-1}\, (1-F(x))^{n-k}.$$

When k = 1, one has $a(x, y) = 0$ because the event $\{C_1 < C_{1:n}\}$ is impossible. Similarly, when k = n one has $b(x, y) = 0$, because the event $\{C_1 > C_{k:n}\}$ is impossible. Using the above formulas, we get

$$I_{k:n} = \int\!\!\int_{\Delta} xy\, a(x,y)\, dx\, dy + \int\!\!\int_{\Delta'} xy\, b(x,y)\, dx\, dy + \int_{D} x^2\, c(x)\, dx$$

It is also convenient to introduce the function H by

$$(3.9) \qquad H(z) = \int_0^z t\, f(t)\, dt \qquad z \geq 0$$

which satisfies $H(+\infty) = E(C) = m$. From the above formulas, we infer

$$I_{k:n} = \frac{(n-1)!}{(k-2)!\,(n-k)!} \int_0^{+\infty} y\, f(y)\, F(y)^{k-2}\, (1-F(y))^{n-k}\, H(y)\, dy$$

$$+ \frac{(n-1)!}{(k-1)!\,(n-k-1)!} \int_0^{+\infty} y\, f(y)\, F(y)^{k-1}\, (1-F(y))^{n-k-1}\, \big(m - H(y)\big)\, dy$$

$$+ \frac{(n-1)!}{(k-1)!\,(n-k)!} \int_0^{+\infty} x^2 f(x)\, F(x)^{k-1} (1-F(x))^{n-k}\, dy \quad .$$

After substituting v = F(y) in the first two integrals and u = F(x) in the third one, we get

$$I_{k:n} = \frac{(n-1)!}{(k-2)!\,(n-k)!} \int_0^1 F^{-1}(v)\, v^{k-2} (1-v)^{n-k} H(F^{-1}(v))\, dv$$

$$+ \frac{(n-1)!}{(k-1)!\,(n-k-1)!} \int_0^1 F^{-1}(v)\, v^{k-1} (1-v)^{n-k-1} \big(m - H(F^{-1}(v))\big)\, dv$$

$$+ \frac{(n-1)!}{(k-1)!\,(n-k)!} \int_0^1 F^{-1}(u)^2\, u^{k-1} (1-u)^{n-k}\, du \quad .$$

Equivalently, since k = n-j+1 we have

$$I_{n-j+1:n} = T_1(n, j) + T_2(n, j) + T_3(n, j)$$

where

$$T_1(n, j) = \frac{(n-1)!}{(n-j-1)!\,(j-1)!} \int_0^1 F^{-1}(v)\, v^{n-j-1} (1-v)^{j-1} H(F^{-1}(v))\, dv$$

$$T_2(n, j) = \frac{(n-1)!}{(n-j)!\,(j-2)!} \int_0^1 F^{-1}(v)\, v^{n-j} (1-v)^{j-2} \big(m - H(F^{-1}(v))\big)\, dv$$

$$T_3(n, j) = \frac{(n-1)!}{(n-j)!\,(j-1)!} \int_0^1 F^{-1}(u)^2\, u^{n-j} (1-u)^{j-1}\, du \quad .$$

According to a previous observation, the first term is absent if j = n and the second one is absent if j = 1. Now, we turn to the computation of

$$(3.10) \quad E(N\, E(C_1\, C_{N-j+1:N})) = \sum_{n\geq j+1} p_n\, T_1(n, j) + \sum_{n\geq j} p_n\, T_2(n, j)$$

$$+ \sum_{n\geq j} p_n\, T_3(n, j) \quad .$$

As to the first summation in (3.10), we get

$$\sum_{n\geq j+1} p_n\, T_1(n, j)$$

$$= \frac{1}{(j-1)!} \int_0^1 F^{-1}(v)\,(1-v)^{j-1}\,H(F^{-1}(v)) \Big(\sum_{n \geq j+1} \frac{n!}{(n-j-1)!} v^{n-j-1} \Big)\, dv$$

$$= \frac{1}{(j-1)!} \int_0^1 F^{-1}(v)\,(1-v)^{j-1}\,H(F^{-1}(v))\,\psi_N^{(j+1)}(v)\, dv$$

where $\psi_N^{(j+1)}$ denotes the $(j+1)^{th}$ derivative of ψ_N, the probability generating function of N. Recall that

$$\psi_N(t) = E(t^N) \qquad 0 \leq t \leq 1.$$

As to the second and third summation in (3.10), we get similarly

$$\sum_{n \geq j} p_n T_2(n, j) = \frac{1}{(j-2)!} \int_0^1 F^{-1}(v)\,(1-v)^{j-2}\,\big(m - H(F^{-1}(v))\big)\,\psi_N^{(j)}(v)\, dv$$

$$\sum_{n \geq j} p_n T_3(n, j) = \frac{1}{(j-1)!} \int_0^1 F^{-1}(u)^2\,(1-u)^{j-1}\,H(F^{-1}(u))\,\psi_N^{(j)}(u)\, dv.$$

Consequently, we have proven the following result.

Theorem 3.2

For each j = 1, ..., p one has

$$E(N\,E_N(C_1\,C_{N-j+1:N})) = \frac{1}{(j-1)!} \int_0^1 F^{-1}(v)\,(1-v)^{j-1}\,H(F^{-1}(v))\,\psi_N^{(j+1)}(v)\, dv$$
$$+ \frac{1}{(j-2)!} \int_0^1 F^{-1}(v)\,(1-v)^{j-2}\,\big(m - H(F^{-1}(v))\big)\,\psi_N^{(j)}(v)\, dv$$
$$+ \frac{1}{(j-1)!} \int_0^1 F^{-1}(u)^2\,(1-u)^{j-1}\,\psi_N^{(j)}(u)\, du.$$

Theorem 3.2 allows us to derive formulas for E(X X") in the LCR and ECOMOR treaties. In view of (3.4), we also need to determine Var(X"). In the LCR(p) treaty we have

$$Var(X") = \sum_{j=1}^{p} Var(C_{N-j+1:N}) + 2\sum_{j=2}^{p} \sum_{i=1}^{j-1} cov(C_{N-i+1:N}\,,\,C_{N-j+1:N}).$$

In the ECOMOR(p) treaty, the following formula holds

$$\mathrm{Var}(X'') = \sum_{j=1}^{p-1} \mathrm{Var}(C_{N-j+1:N}) \quad + (p-1)^2\, \mathrm{Var}(C_{N-p+1:N})$$

$$+\ 2\sum_{j=2}^{p-1} \sum_{i=1}^{j-1} \mathrm{cov}(C_{N-i+1:N}\ ,\ C_{N-j+1:N})$$

$$-\ 2(p-1) \sum_{i=2}^{p-1} \quad \mathrm{cov}(C_{N-i+1:N}\ ,\ C_{N-p+1:N}).$$

General formulas allowing for evaluating

$$E(C_{N-j+1:N})\ \mathrm{Var}(C_{N-j+1:N}) \qquad \text{and} \qquad \mathrm{cov}(C_{N-i+1:N}\ ,\ C_{N-j+1:N})$$

have been recalled in the beginning of this section. Using these formulas and the above relationships, and returning to (3.4), it is now possible to derive formulas for Var(X') in the LCR and ECOMOR treaties. These formulas will be used in Section 5, as well as formulas of Theorems 3.1 and 3.2.

4. Comparison of LCR and ECOMOR reinsurance with Excess-of-Loss Reinsurance

It is interesting to compare the efficiency of LCR and ECOMOR treaties with the excess of loss (XL) treaty that is frequently used in the reinsurance world. For this purpose, we consider an XL treaty whose priority is denoted by s and whose upper limit is $+\infty$.

The comparison will be done along the following lines. Generally speaking insurance and reinsurance premiums include a safety loading that is added to the pure premium in order to increase insurers' solvency. More precisely, if X stands for the aggregate claim amount relative to a group of risks during a given time period, the loaded premium $\Pi(X)$ will have the following form

(4.1) $\qquad \Pi(X) = E(X) + SL(X)$

where SL(X) denotes the safety loading. This loading is chosen in reference to some premium principle. For example, according to the expectation principle, the safety loading will be

$$SL(X) = \alpha\, E(X)$$

where α is a suitable positive coefficient. If the insurer has chosen the standard deviation principle, the safety loading will read as

$$SL(X) = \alpha\ \sigma(X).$$

This is also valid in the framework of a reinsurance treaty. In a similar way, the reinsurance premium, denoted by $\Pi_r(X")$, has the following form

$$\Pi_r(X") = E(X") + SL_r(X") \tag{4.2}$$

where X" denotes the reinsurer share, as above, and $SL_r(X")$ denotes the corresponding safety loading. If the reinsurer applies the expectation principle (resp. standard deviation principle), the safety loading will be given by $SL_r(X") = \beta\ E(X")$ (resp. $SL_r(X") = \beta\ \sigma(X")$), where β is a positive coefficient (possibly different from α).

Further, we consider the underwriting profit (profit for short) of the insurer, i.e. not including financial benefits. In the absence of reinsurance, it is obviously equal to

$$\Pi(X) - X.$$

After a reinsurance arrangement has taken place, the claim expenses are divided according to the usual formula, i.e.

$$X = X' + X"$$

where X' (resp. X") stands for the reinsured (resp. reinsurer) share. Then, the insurer profit, denoted by Z, becomes

$$Z = \Pi(X) - X' - \Pi_r(X").$$

In view of (4.1) and (4.2), it is readily seen that its first two moments are given by

$$E(Z) = SL(X) - SL_r(X") \quad \text{and} \qquad \sigma(Z) = \sigma(X'). \tag{4.3}$$

In particular, if both the cedant and the reinsurer use the expectation principle (case I), we get

$$E(Z) = \alpha\ E(X) - \beta\ E(X").$$

If the standard deviation principle is used (case II), we get

$$E(Z) = \alpha\ \sigma(X) - \beta\ \sigma(X").$$

The comparison between XL reinsurance and LCR or ECOMOR reinsurance will be done as follows. For example, consider the LCR(p) treaty and the XL treaty with priority s, denoted by XL(s). If the XL treaty is in force, we can express X as

$$X = Y'(s) + Y''(s)$$

where $Y'(s)$ (resp. $Y''(s)$) denotes the reinsured (resp. the reinsurer) share in the XL(s) treaty. Recall that

$$Y'(s) = \sum_{i=1}^{N} C_i'(s) \quad \text{and} \qquad Y''(s) = \sum_{i=1}^{N} C_i''(s)$$

where $C_i'(s)$ and $C_i''(s)$ are given by

$$C_i'(s) = \min(C_i\,, s) \ \text{and} \qquad C_i''(s) = \max(0, C_i\ - s)$$

for all i = 1, ..., N. As for the LCR(p) treaty, we can write

$$X = X'(p) + X''(p)$$

where the variable p refers to the LCR(p) treaty, i.e. to the number of claims paid by the reinsurer.

We begin by computing the priority, say $s^{\#}$, such that the expectation of the insured profit takes on the same value after the applicaton of both treaties. Namely, $s^{\#}$ satisfies the equation

(4.4) $$SL(X) - SL_r(X''(p)) = SL(X) - SL_r(Y''(s^{\#}))$$

or equivalently

(4.5) $$SL_r(X''(p)) = SL_r(Y''(s^{\#})).$$

Then, we compare the standard deviations of the reinsured share in the LCR(p) treaty and in the XL($s^{\#}$) treaty, namely $\sigma(X'(p))$ and $\sigma(Y'(s))$.

In case I, equation (4.5) becomes $E(X''(p)) = E(Y''(s^{\#}))$. and in case II, $\sigma(X''(p)) = \sigma(Y''(s^{\#}))$. Further, if the number of claims is Poisson distributed with parameter λ, formulas (2.3), applied to $Y''(s^{\#})$, yields

(4.6) $$E(X''(p)) = \lambda\, E(C''(s^{\#})) \qquad \text{(case I)}$$

and

(4.7) $$\sigma(X''(p)) = \sqrt{\lambda\, E(C''(s^{\#})^2)} \qquad \text{(case II).}$$

Of course, the value of the root $s^{\#}$ is not the same in (4.6) and in (4.7).

Remark 4.1

When the safety loading is based on the expectation principle, pure premiums for LCR (or ECOMOR) and XL treaties are equal, as shown by equation (4.6). This is no longer true when the safety loading is based on the standard deviation principle. In this case, the equality holds between the standard deviation of the reinsurer share relative to LCR or ECOMOR and XL treaties. This is expressed by equation (4.7).

Remark 4.2

The following ratio, denoted by T is often used as a simple measure of solvency

$$T = \frac{R + E(Z)}{\sigma(Z)}$$

where R stands for the amount of the risk reserve and Z for the cedant's profit as defined above. The ratio T, referred to as the solvency coefficient, is closely connected to the ruin probability. Higher values of T correspond lower values of ruin probability, thus to better solvency level. According to equation (4.3) we have

$$T = \frac{R + SL(X) - SL_r(X'')}{\sigma(X')} \quad . \tag{4.8}$$

This is valid for any reinsurance treaty. In particular, when the safety loading is based on the expectation principle, (4.8) becomes

$$T = \frac{R + \alpha\, E(X) - \beta\, E(X'')}{\sigma(X')} \tag{4.9}$$

and when it is based on the standard deviation principle

$$T = \frac{R + \alpha\, \sigma(X) - \beta\, \sigma(X'')}{\sigma(X')} \quad . \tag{4.10}$$

In (4.9) and (4.10), the loading coefficients α and β are then involved, as one could expect.

On the other hand, consider for example the LCR(p) and the XL(s) treaties. Then, the solvency coefficient T is given by

$$T_{LCR}(p) = \frac{R + SL(X) - SL_r(X''(p))}{\sigma(X'(p))} \tag{4.11}$$

for the LCR(p) treaty and by

$$T_{XL}(s) = \frac{R + SL(X) - SL_r(Y''(s))}{\sigma(Y'(s))} \tag{4.12}$$

for the XL(s) treaty. Observe that, from equation (4.4), the numerators in (4.11) and (4.12) are equal, which implies

$$\frac{T_{LCR}(p)}{T_{XL}(s)} = \frac{\sigma(Y'(s))}{\sigma(X'(p))} .$$

In particular, the inequality $\sigma(X'(p)) \leq \sigma(Y'(s))$ is equivalent to $T_{LCR}(p) \geq T_{XL}(s)$.

Remark 4.3

In order to compare XL and LCR (or ECOMOR) treaties we have considered the cedant's underwriting profit and we have chosen the priority s of the XL treaty so that the profit is unchanged. However, the comparison could have been based on other criteria. Let us cite for example the loaded premium ratio

$$\frac{\Pi(X) - \Pi_r(X'')}{\Pi(X)} = 1 - \frac{\Pi_r(X'')}{\Pi(X)}$$

which represents the fraction of the premium retained by the cedant.

5. Numerical Examples

Numerical examples are given below for the LCR(p) and the ECOMOR(p) treaties, when the number of claims is Poisson distributed with parameter $\lambda = 40$ and the individual claim distribution is either translated exponential or generalized Pareto. These distributions are strongly different in that the translated exponential has a light tail, whereas the generalized Pareto has an heavy tail. Further, in view of the comparison with the XL treaty, we have considered two kinds of safety loading, namely the safety loading based on the expectation principle (case I) and the safety loading based on the standard deviation principle (case II).

When the claim size C is translated exponential distributed, its cumulative distribution function F is given by

$$F(x) = 1 - \exp\big(\alpha\,(x - x_0)\big) \text{ if } x \geq x_0 \text{ , } 0 \qquad \text{if } x < x_0$$

where the parameters x_0 and α are positive. Here we have chosen

$$x_0 = 500 \qquad \text{and} \qquad \alpha = 0{,}01.$$

In this case, it follows from (2.1) and (2.2) that E(X) = 24 000 and $\sigma(X)$ = 3847,08. As to the generalized Pareto, the cumulative distribution function is given by

$$F(x) = 1 - \left(\frac{x_0 + b}{x + b}\right)^{\alpha} \quad \text{if } x \geq x_0 \text{ , } 0 \quad \text{otherwise,}$$

where the parameters must satisfy $x_0 > 0$, $\alpha > 0$ and $b > -x_0$. Here we have chosen $x_0 = 100$, $b = 500$ and $\alpha = 2.5$, which entails $E(X) = 20\ 000$ and $\sigma(X) = 6480{,}74$.

Eight tables are presented hereafter in order to illustrate the previous discussion. In Tables 1, 1 bis, 3 and 3 bis, the claim size distribution is the translated exponential, in the other ones it is the generalized Pareto. In Tables 1 to 2 bis it is assumed that both the cedant and the reinsurer apply the pure premium principle, whereas in Tables 3 to 4 bis, the cedant and the reinsurer are assumed to use the standard deviation principle. Further, Tables 1, 2, 3 and 4 concern LCR treaties, whereas Tables 1 bis, 2 bis, 3 bis and 4 bis concern ECOMOR treaties.

In each table, the first column shows the values of p, the number of the largest claims involved in LCR or ECOMOR treaty, the second one shows the corresponding values of the pure premium retained by the cedant. The third column displays the standard deviation of the cedant's share, according to the reinsurance arrangement (LCR or ECOMOR). The fourth column displays the priority $s^{\#}$ of the XL treaty that produces the same profit for the cedant (taking into account the principle that has been chosen for calculating the safety loading). The fifth column contains the values of the pure premium ratio (PPR), namely

$$\frac{E(X')}{E(X)} \quad .$$

The sixth column shows the standard deviation ratio relative to the LCR or ECOMOR treaty, namely

$$\frac{\sigma(X'_{LCR})}{\sigma(X)} \qquad \text{or} \qquad \frac{\sigma(X'_{ECO})}{\sigma(X)} \quad .$$

In the tables, these ratios are also denoted by SDR-LCR and SDR-ECO. The last column shows the standard deviation ratio relative to the $XL(s^{\#})$ treaty. It is denoted by SDR-XL and defined by

$$\frac{\sigma(X'_{XL})}{\sigma(X)} \quad .$$

Each table is followed by a figure that displays the data of the last two column and allows us to compare visually the efficiency of LCR or ECOMOR treaty with the XL one.

TABLE 1. LCR vs XL for the translated exponential safety loading based on the expectation principle

p	E(X'(p))	σ(X'(p))	priority	PPR	SDR-LCR	SDR-XL
1	23 073	3 822	646,25	0,961	0,994	0,952
2	22 247	3 801	582,48	0,927	0,988	0,916
3	21 470	3 780	545,81	0,895	0,983	0,883
4	20 727	3 760	520,06	0,864	0,977	0,852
5	20 009	3 741	500,22	0,834	0,972	0,822
6	19 310	3 723	482,76	0,805	0,968	0,794
7	18 629	3 704	465,72	0,776	0,963	0,766
8	17 961	3 686	449,04	0,748	0,958	0,738
9	17 307	3 668	432,66	0,721	0,954	0,711
10	16 663	3 651	416,57	0,694	0,949	0,685

FIGURE 1. LCR vs XL for translated exponential - safety loading based on the expectation

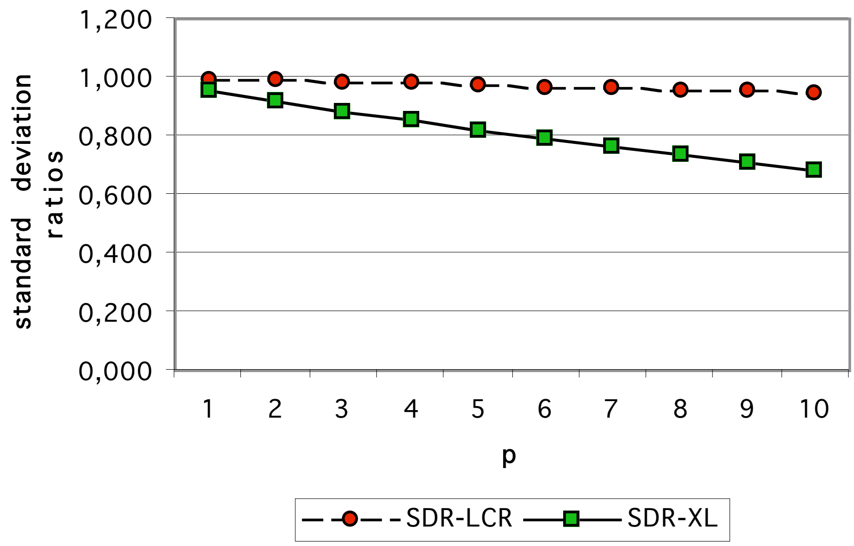


TABLE 1 bis. ECOMOR vs XL for the translated exponential safety loading based on the expectation principle

p	E(X'(p))	σ(X'(p))	priority	PPR	SDR-ECO	SDR-XL
1	0	0	0	0	0	0
2	23 900	3 846	868,89	0,996	1,000	0,993
3	23 800	3 844	799,57	0,992	0,999	0,988
4	23 700	3 843	759,03	0,988	0,999	0,982
5	23 600	3 842	730,26	0,983	0,999	0,977
6	23 500	3 841	707,94	0,979	0,998	0,972
7	23 400	3 839	689,71	0,975	0,998	0,967
8	23 300	3 838	674,30	0,971	0,998	0,963
9	23 200	3 837	660,94	0,967	0,997	0,958
10	23 100	3 835	649,17	0,962	0,997	0,953

FIGURE 1 bis. ECOMOR vs LCR for translated exponential - safety loading based on the expectation

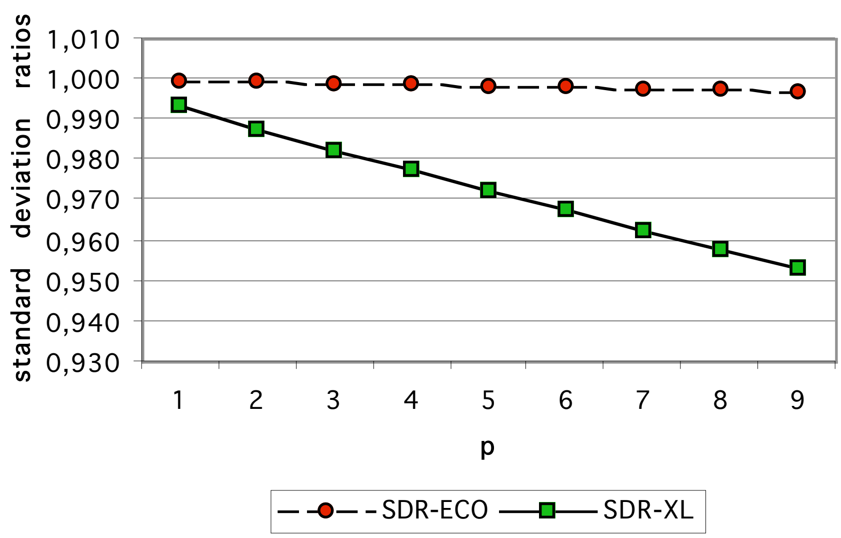


TABLE 2. LCR vs XL for the Generalized Pareto $\alpha = 2{,}5$
safety loading based on the expectation principle

p	E(X'(p))	σ(X'(p))	priority	PPR	SDR-LCR	SDR-XL
1	16 592	4 214	1 182,36	0,830	0,650	0,512
2	14 748	3 720	760,84	0,737	0,574	0,423
3	13 372	3 412	579,70	0,669	0,526	0,368
4	12 246	3 180	472,50	0,612	0,491	0,328
5	11 283	2 991	399,48	0,564	0,462	0,296
6	10 437	2 830	345,62	0,522	0,437	0,269
7	9 681	2 689	303,78	0,484	0,415	0,247
8	8 996	2 563	270,09	0,450	0,395	0,227
9	8 371	2 449	242,23	0,419	0,378	0,209
10	7 796	2 344	218,72	0,390	0,362	0,194

FIGURE 2. LCR vs XL for the generalized Pareto - safety loading based on the expectation

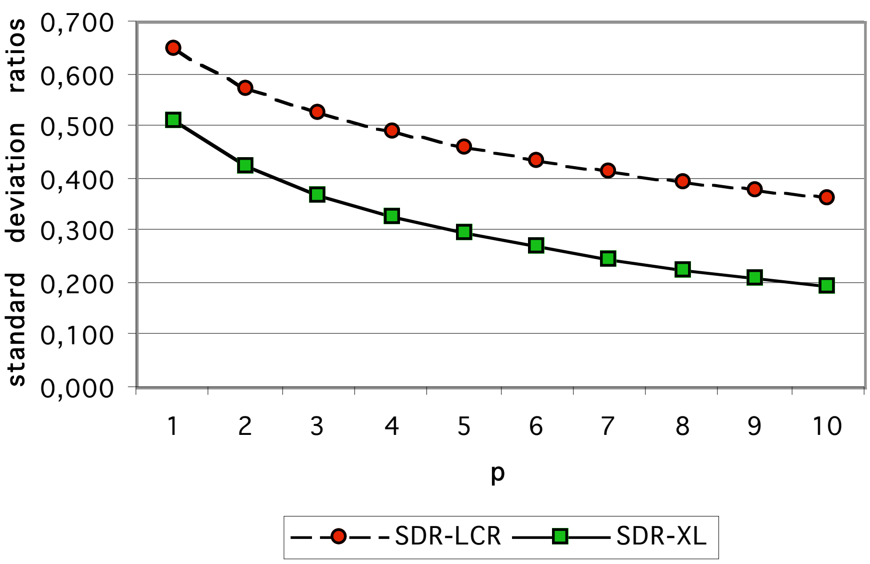


TABLE 2 bis. ECOMOR vs XL for the Generalized Pareto $\alpha = 2{,}5$
safety loading based on the expectation principle

p	E(X'(p))	σ(X'(p))	priority	PPR	SDR-ECO	SDR-XL
1	0	0	0	0	0	0
2	18 437	4 829	2 328,62	0,922	0,745	0,637
3	17 499	4 459	1 567,73	0,875	0,688	0,566
4	16 749	4 230	1 235,93	0,837	0,653	0,521
5	16 099	4 058	1 037,25	0,805	0,626	0,486
6	15 513	3 919	900,49	0,776	0,605	0,457
7	14 975	3 800	798,57	0,749	0,586	0,433
8	14 472	3 695	718,63	0,724	0,570	0,411
9	13 999	3 602	653,62	0,700	0,556	0,392
10	13 548	3 517	599,32	0,677	0,543	0,375

FIGURE 2 bis. ECOMOR vs XL for generalized Pareto - safety loading based on the expectation

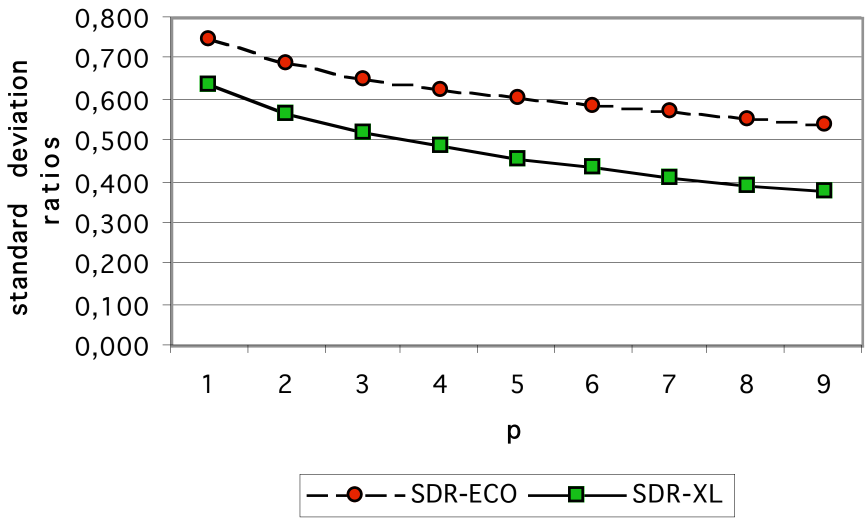


TABLE 3. LCR vs XL for the translated exponential safety loading based on the standard deviation principle

p	E(X'(p))	σ(X'(p))	priority	PPR	SDR-LCR	SDR-XL
1	23 073	3 822	888,43	0,961	0,994	0,994
2	22 247	3 801	810,67	0,927	0,988	0,989
3	21 470	3 780	766,74	0,895	0,983	0,984
4	20 727	3 760	736,16	0,864	0,977	0,978
5	20 009	3 741	712,73	0,834	0,972	0,973
6	19 310	3 723	693,73	0,805	0,968	0,969
7	18 629	3 704	677,76	0,776	0,963	0,964
8	17 961	3 686	663,98	0,748	0,958	0,959
9	17 307	3 668	651,88	0,721	0,954	0,954
10	16 663	3 651	641,08	0,694	0,949	0,950

FIGURE 3. LCR vs XL for the transated exponential - safety loading on the standard deviation

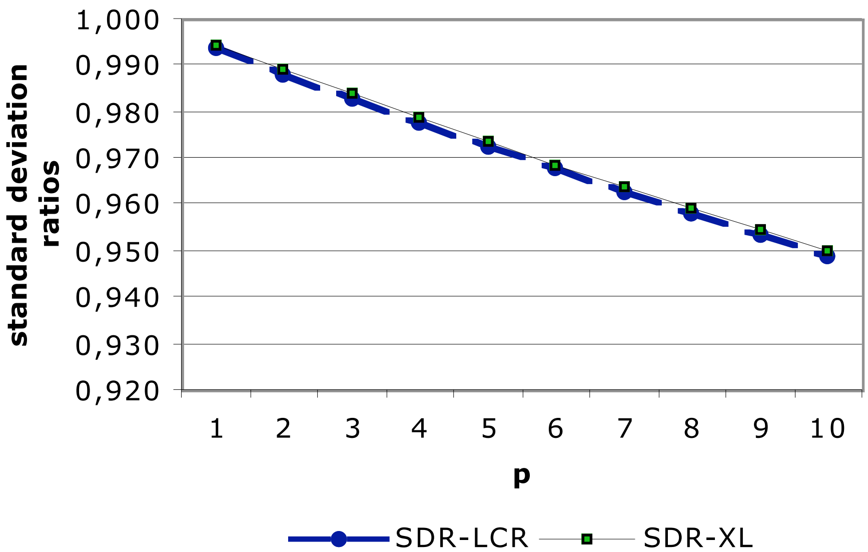


TABLE 3 bis. ECOMOR vs XL for the translated exponential safety loading based on the standard deviation principle

p	E(X'(p))	σ(X'(p))	priority	PPR	SDR-ECO	SDR-XL
1	0	0	0	0	0	0
2	23 900	3 846	938,20	0,996	1,000	0,996
3	23 800	3 844	868,89	0,992	0,999	0,993
4	23 700	3 843	828,34	0,988	0,999	0,991
5	23 600	3 842	799,57	0,983	0,999	0,988
6	23 500	3 841	777,26	0,979	0,998	0,985
7	23 400	3 839	759,03	0,975	0,998	0,982
8	23 300	3 838	743,61	0,971	0,998	0,980
9	23 200	3 837	730,26	0,967	0,997	0,977
10	23 100	3 835	718,48	0,962	0,997	0,975

FIGURE 3 bis. ECOMOR vs XL for the translated exponential - safety loading based on the standard deviation

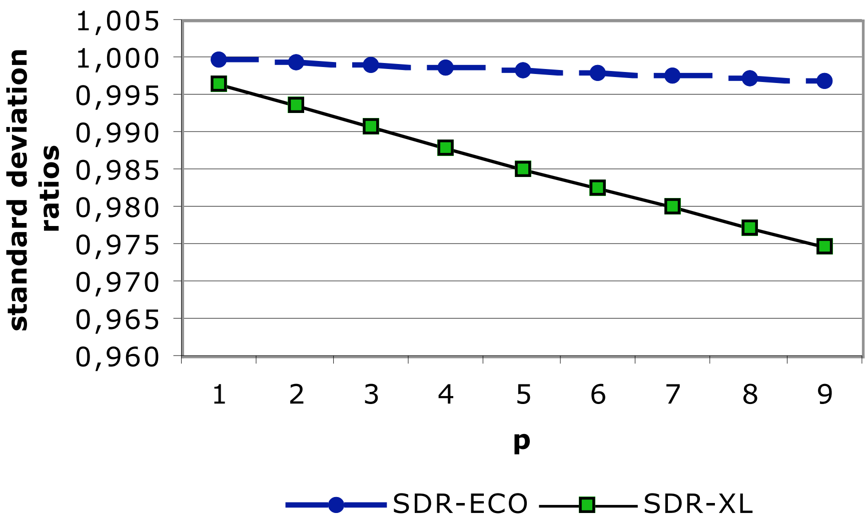


TABLE 4. LCR vs XL for the Generalized Pareto $\alpha = 2,5$
safety loading based on the standard deviation principle

p	E(X'(p))	σ(X'(p))	priority	PPR	SDR-LCR	SDR-XL
1	16 592	4 214	2 813,31	0,830	0,650	0,668
2	14 748	3 720	1 730,65	0,737	0,574	0,585
3	13 372	3 412	1 323,95	0,669	0,526	0,534
4	12 246	3 180	1 094,60	0,612	0,491	0,497
5	11 283	2 991	941,79	0,564	0,462	0,466
6	10 437	2 830	830,22	0,522	0,437	0,441
7	9 681	2 689	743,94	0,484	0,415	0,419
8	8 996	2 563	674,48	0,450	0,395	0,399
9	8 371	2 449	616,93	0,419	0,378	0,381
10	7 796	2 344	568,16	0,390	0,362	0,364

FIGURE 4. LCR vs XL for the generalized Pareto - safety loading based on the standard deviation

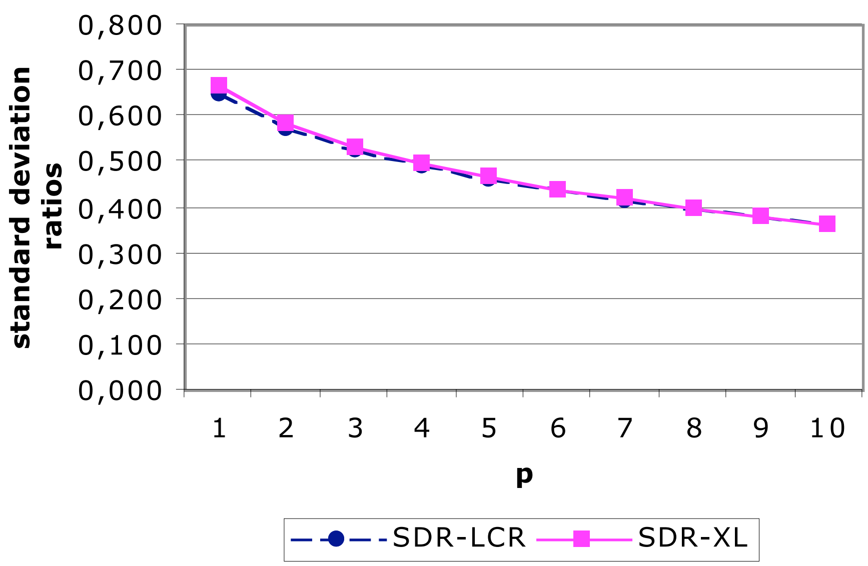
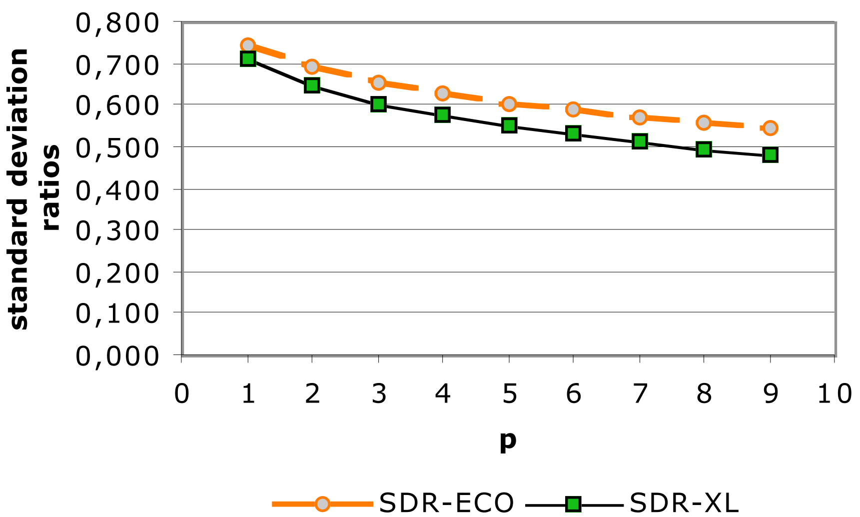


TABLE 4 bis. ECOMOR vs LCR for the Generalized Pareto $\alpha = 2{,}5$
safety loading based on the standard deviation principle

p	E(X'(p))	σ(X'(p))	priority	PPR	SDR-ECO	SDR-XL
1	0	0	0	0	0	0
2	18 437	4 829	3 757,13	0,922	0,745	0,711
3	17 499	4 459	2 439,66	0,875	0,688	0,645
4	16 749	4 230	1 924,70	0,837	0,653	0,604
5	16 099	4 058	1 629,00	0,805	0,626	0,574
6	15 513	3 919	1 429,94	0,776	0,605	0,549
7	14 975	3 800	1 283,64	0,749	0,586	0,528
8	14 472	3 695	1 169,97	0,724	0,570	0,510
9	13 999	3 602	1 078,15	0,700	0,556	0,494
10	13 548	3 517	1 001,87	0,677	0,543	0,479

FIGURE 4 bis. ECOMOR vs LCR for the generalized Pareto - safety loading based on standard deviation



In the above examples, it can be seen that the XL treaty is always more efficient than the ECOMOR treaty, in that it produces a smaller standard deviation for the cedant for the same cost. In addition, the XL treaty is often more efficient than the LCR treaty. However, when the safety loading is based on the standard deviation principle, the LCR treaty is a little better than the XL treaty. But it is worthwhile to note that we have assumed that the upper limit of the XL treaty is infinite, which is not realistic. Thus, the efficiency of the XL treaty might be better in the presence of a finite upper limit. On the other hand, it can be seen that the SDR-gap between the LCR (or ECOMOR) treaty and the XL treaty increases with the number p of claims in consideration.

It can be also observed that in all treaties the variance reduction is higher for the generalized Pareto than for the translated exponential. This must be connected to the fact that the generalized Pareto has a heavier tail than the translated exponential.

As to the numerical computations, they have been performed using the formulas presented in Section 3, with the Maple 9.5 software. The Monte-Carlo method has also been used for the translated exponential distribution. As for the generalized Pareto distribution with $\alpha = 2.5$, the Monte Carlo method was available for the expectation, but not for the standard deviation. Indeed, although the theoretical variance Var(C) exists, the variance of its estimator, namely the sample variance, involves the moment of order 4 of the distribution, which is infinite. Thus, the convergence of the sample variance in the Monte Carlo algorithm is not good.

The numerical results given by Berglund for the moments of the reinsurer's share show that replacing the Poisson distribution by the negative binomial distribution does not produce major changes. A similar behavior can be conjectured for the cedant's share.

6. Conclusion

In this work, we have derived formulas allowing one to calculate the expectation and, more importantly, the variance of the cedant's share in the framework of the LCR or ECOMOR treaty. We have also compared the efficiency of such treaties with the XL treaty by some numerical examples, under two different hypotheses on the safety loading. In these examples, we have observed that the XL treaty always dominates the ECOMOR treaty. For the LCR treaty the situation is more complex. In some cases, the LCR treaty is a little better than the XL treaty, especially when the safety loading is based on the standard deviation.

Of course, other examples could be considered in order to get more information. Especially, the case of XL treaties with finite upper limit could be of interest.

Further, a natural question arising from the present paper is that of the optimality of a reinsurance treaty among a given class of treaties from the cedant's point of view. In view of the above numerical examples, one could also ask the following simpler question: is it possible to find conditions under which a LCR treaty is better than an XL treaty ? A theoretical result on the comparison of LCR and ECOMOR treaties is also strongly suggested by the numerical results.

On the other hand, we have only considered moments or order 1 and 2. However, it is known that the real aggregate claim amount distributions as well as the claim size distribution are seldom symmetric. Thus, the introduction of moments of higher order, especially of order 3, could be pertinent.

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