ECONOMIC RISK CAPITAL OF GUARANTEED CASH-FLOWS

UNDER FRÉCHET-MARKOV RETURN MODELS

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Abstract

The multi-period economic risk capital of a guaranteed cash-flow under two simple bi- and triatomic Markov chain models of the return is evaluated. It is identified with the conditional value-at-risk of the cash-flow risk given some confidence level. A main application concerns the risk-adjusted return on capital used as a tool of decision making in performance measurement. The obtained results are illustrated and discussed at a real life example.

Key words

value-at-risk, conditional value-at-risk, RAROC, biatomic Markov chain, triatomic Markov chain, random walk, Fréchet copula, skewness, kurtosis

1. Introduction.

The present paper is devoted to the evaluation of the multi-period economic risk capital (ERC) of a guaranteed cash-flow under two simple bi- and triatomic Fréchet-Markov models of the return. A main application concerns the value of the economic risk capital associated to the **r**isk-**a**djusted **r**eturn **o**n **c**apital (RAROC) used as a tool of decision making in performance measurement. In general, the value of the economic risk capital associated to some future random loss is identified with the conditional value-at-risk (CVaR) of the loss given some confidence level α. Mathematically, this is defined as expected value of the "α-tail transform", and can be viewed as a convex combination of the usual value-at-risk (VaR) and the upper conditional value-at-risk (CVaR+). The latter quantity is defined to be the conditional expected loss given the loss strictly exceeds its value-at-risk and represents the "average of the $100(1-\alpha)\%$ worst losses" in a random sample of losses.

As pointed out in recent studies the popular VaR measure, used to assess capital requirements in the insurance and finance industry, suffers from various shortcomings. For example, numerical instability and difficulties occur for non-normal loss distributions, especially in the presence of "fat tails" or/and empirical discreteness. Furthermore, VaR is not a coherent measure of risk in the sense of Arztner et al.(1997/99), and it does not take into account the severity of an incurred adverse loss event. The alternative CVaR measure has some significant advantages over the VaR measure. It is able to quantify dangers beyond value-at-risk and it is coherent. Moreover, it provides a numerical efficient and stable tool in optimization problems under uncertainty. Some recent studies presenting these advantages and further desirable properties include Acerbi(2001), Acerbi and Tasche(2001a/b), Bertsimas et al.(2001), Hürlimann(2001/03), Kusuoka(2001), Pflug(2000), Rockafellar and Uryasev(2000/2001), Testuri and Uryasev(2000), Wirch and Hardy(1999), Yamai and Yoshiba(2001a/b), Yoshiba and Yamai(2001). A more detailed outline of the content follows.

Section 2 introduces our multi-period ERC valuation model for guaranteed discrete-time cash-flows under random returns. To protect the random cash-flow against adverse returns below a deterministic guaranteed return, an investor is supposed to buy in each period a put option. This generates some cost of the guarantee at the finite time horizon. The *cash-flow risk* is defined to be the negative of the possible random gain generated by the protected cash-flow in excess of the sum of the guaranteed cash-flow value and the cost of the guarantee. The multi-period ERC of a guaranteed cash-flow is set equal to the CVaR of the cash-flow risk given some confidence level. A main application concerns the important notion of **r**isk-**a**djusted **r**eturn **o**n **c**apital (RAROC) of a guaranteed cash-flow. Used as a tool in risk-adjustment performance measurement, the RAROC criterion tells us when a guaranteed cash-flow under random returns is preferred to another one.

In Section 3 we consider the following biatomic Markov chain return model. In each period the random return is uniquely identified as a biatomic random variable with fixed mean, standard deviation and skewness. The random returns in the multi-period of valuation are assumed to be identically distributed and follow a one-parameter Markov chain with Fréchet copula bivariate dependence structure, called for short Fréchet-Markov model, which includes as extreme cases independent and comonotone returns. The cost of guarantee is determined under the usual risk-neutral valuation method. Then, it is shown how to evaluate numerically the CVaR risk measure.

Section 4 contains a detailed study of the often encountered guaranteed annuity-due with constant payments under our biatomic Fréchet-Markov return model. A remarkable feature of the put option strategy used to protect the random annuity is the *constant* amount of *ERC* as long as the chosen confidence level is sufficiently high. Since ERC is bounded above by the cost of guarantee, we maximize this quantity under variation of the skewness parameter. This yields prudent ERC upper bounds. The obtained results are illustrated and discussed at the real life example of the Fed Funds rates of return over the period 1988-1997.

Section 5 extends the analysis using a similar triatomic Fréchet-Markov chain return model. Based on the solution of the algebraic moment problem of order three, we first identify the one-period random returns as a one-parameter family of triatomic random variables with fixed mean, standard deviation, skewness and kurtosis. The remaining parameter is uniquely determined by the VaR condition that a return less than the smallest possible atom occurs only with the loss probability $\varepsilon = 1-\alpha$, where α is the confidence level used to calculate CVaR. To determine the cost of guarantee, we replace the probabilities by risk-neutral probabilities. The numerical evaluation of the CVaR risk measure is similar to the one presented in Section 3. A comparative numerical illustration follows for the guaranteed annuity-due with constant payments.

2. Multi-period ERC valuation model.

A discrete-time *cash-flow* over the time horizon $[0,T]$, T a positive integer, is a sequence of payments $\boldsymbol{c} = (c_1,...,c_T)$, where the payment c_k is due at time $k-1, k=1,...,T$. The payments may be strictly positive (long position), strictly negative (short position) or zero (vanishing cash payment).

For simplicity suppose that the return in the k-th period $[k-1,k]$ is a random variable $R_k, k=1,...,T$. We assume that $R_1, R_2, ..., R_T$ are identically distributed with finite expected return $r = E[R_k]$ and standard deviation $\sigma = \sqrt{Var[R_k]}$. Over the time horizon $[0,T]$, the payments should yield the deterministic guaranteed return r_g. The corresponding *guaranteed cash-flow* induces a liability, whose accumulated value at time T is given by

$$L_T = \sum_{k=1}^{T} c_k \cdot (1+r_g)^{T-k+1}. \tag{2.1}$$

To obtain the guaranteed return with certainty, an investor will buy on the financial market in each period $[k-1,k]$ a certain amount of put options with payoff $(r_g - R_k)_+$. According to the put-call parity relation, the total random return in period $[k-1,k]$ satisfies the identity

$$R_k + (r_g - R_k)_+ = r_g + (R_k - r_g)_+. \tag{2.2}$$

The random accumulated value at time T of the *protected cash-flow* is then given by

$$V_T = \sum_{k=1}^{T} c_k \cdot \prod_{j=k}^{T} \left[1 + r_g + (R_k - r_g)_+\right]. \tag{2.3}$$

It remains to determine the *cost of guarantee*. Denote by $P(r_g)$ the put option price, to be paid at time $k-1$, which yields in each period $[k-1,k]$ the payoff $(r_g - R_k)_+, k=1,...,T$. In each period $[t-1,t], t=1,...,T$, the accumulated amount of the sequence of payments $c_1,...,c_t$, that is $\sum_{k=1}^{t} c_k (1+r_g)^{t-k+1}$, must be guaranteed at time t by buying put options with payoff $(r_g - R_t)_+$. Using (2.2) the cost of this guarantee at time $t-1$ equals $K_{t-1} = P(r_g) \cdot \sum_{k=1}^{t} c_k (1+r_g)^{t-k+1}, t = 1,...,T$. Valued at the risk-free return r_f, the accumulated value at time T of the total cost of guarantee is then equal to

$$C_T = \sum_{t=0}^{T-1} K_t (1+r_f)^{T-t}. \tag{2.4}$$

It will be assumed throughout that the risk-free return, the guaranteed return and the expected return satisfy the inequality

$$r_g < r_f < r. \tag{2.5}$$

The fact that $r_g < r_f$ follows from divers arguments (e.g. Devolder(1986/91), Kozik(1991), Hürlimann(1991a/b/96), Wilkie(1991). The above protected cash-flow should be compared with the guaranteed cash-flow under the cost of guarantee. The generated aggregate surplus at time T, defined as difference between assets and liabilities, is denoted and equal to

$$G_T = V_T - L_T - C_T. \tag{2.6}$$

It represents the possible random gain of the protected cash-flow in excess of the sum of the guaranteed cash-flow value and the cost of guarantee. Its negative value is called *cash-flow risk*. The *economic risk capital* (ERC) associated to the cash-flow risk $X_T = -G_T = C_T + L_T - V_T$ will be identified with the conditional value-at-risk to some confidence level α defined by Rockafellar and Uryasev(2001).

Given a loss random variable X with distribution function $F_X(x) = \Pr(X \le x)$, consider the *value-at-risk* (VaR) to the confidence level α, defined as the lower α-quantile

$$VaR_\alpha[X] = Q_X^\ell(\alpha) = \inf\{x : F_X(x) \ge \alpha\}, \tag{2.7}$$

and the *upper conditional value-at-risk* ($CVaR^+$) to the confidence level α, defined by

$$CVaR_\alpha^+[X] = E[X | X > VaR_\alpha[X]]. \tag{2.8}$$

The VaR quantity represents the maximum possible loss, which is not exceeded with the probability α (in practice $\alpha = 95\%, 99\%, 99.75\%$). The $CVaR^+$ quantity is the conditional expected loss given the loss strictly exceeds its value-at-risk. Next, consider the *α-tail transform* X^α of X with distribution function

$$F_{X^\alpha}(x) = \begin{cases} 0, & x < VaR_\alpha[X], \\ \dfrac{F_X(x) - \alpha}{1-\alpha}, & x \ge VaR_\alpha[X]. \end{cases} \tag{2.9}$$

Rockafellar and Uryasev(2001) define *conditional value-at-risk* (CVaR) to the confidence level α as expected value of the α-tail transform, that is by

$$CVaR_{\alpha}[X] = E[X^{\alpha}]. \tag{2.10}$$

The obtained measure is a coherent risk measure in the sense of Arztner et al.(1997/99) and coincides with $CVaR^{+}$ in the case of continuous distributions. As pointed out by Hürlimann(2003), several equivalent formulas exist for the evaluation of (2.10). Perhaps the simplest and most practical one is the following stop-loss transform representation.

Lemma 2.1. One has the formula

$$CVaR_{\alpha}[X] = VaR_{\alpha}[X] + \frac{1}{\varepsilon} \cdot \pi_X(VaR_{\alpha}[X]), \tag{2.11}$$

where $\pi_X(x) = E[(X - x)_+]$ is the stop-loss transform, and $\varepsilon = 1 - \alpha$ is interpreted as loss probability.

Proof. By definition (2.10) one has $CVaR_{\alpha}[X] = E[X^{\alpha}] = \int_0^{\infty} \overline{F}_{X^{\alpha}}(x)dx - \int_{-\infty}^{0} F_{X^{\alpha}}(x)dx$. Using (2.9) one obtains distinguishing between the two cases $VaR_{\alpha}[X] \geq 0$ and $VaR_{\alpha}[X] < 0$ without difficulty the desired expression (2.11). ◊

In the special situation of discrete loss distributions with ordered support $x_1 < x_2 < ... < x_k < x_{k+1} < ...$, which will be used to evaluate our CVaR bounds in the present paper, numerical evaluation proceeds as follows. Let $f_k = \Pr(X = x_k)$ denote the probability that the loss takes the value x_k, $k = 1,2,3,...$, and assume the finite mean $\mu_X = E[X]$ is known. Determine the unique index k_{α} such that

$$\sum_{k=1}^{k_{\alpha}-1} f_k < \alpha \leq \sum_{k=1}^{k_{\alpha}} f_k . \tag{2.12}$$

Then one has

$$VaR_{\alpha}[X] = x_{k_{\alpha}},$$

and one obtains from Lemma 2.1 that

$$\begin{aligned} CVaR_{\alpha}[X] &= VaR_{\alpha}[X] + \frac{1}{\varepsilon} \cdot \{\mu_X - VaR_{\alpha}[X] + E[(VaR_{\alpha}[X] - X)_+]\} \\ &= \frac{1}{\varepsilon} \cdot \left\{ \mu_X - \alpha \cdot x_{k_{\alpha}} + \sum_{k=0}^{k_{\alpha}} (x_{k_{\alpha}} - x_k) \cdot f_k \right\}. \end{aligned} \tag{2.13}$$

In particular, the loss probabilities must only be evaluated up to the index k_{α} satisfying the inequality (2.12).

The CVaR risk measure is of great importance in decision making because it can be used as a tool in risk-adjusted performance measurement. Consider the random gain (2.6) per unit of conditional value-at-risk, called *CVaR gain ratio*, which is defined by

$$\frac{G_T}{CVaR_\alpha[X_T]}. \tag{2.14}$$

The expected value of the CVaR gain ratio measures the **r**isk-**a**djusted **r**eturn **o**n **c**apital. This way of computing the return is commonly called RAROC (e.g. Matten(1996), p.59), and is defined by

$$RAROC_\alpha[G_T] = \frac{E[G_T]}{CVaR_\alpha[X_T]}. \tag{2.15}$$

Now, if an investor has to decide upon the more profitable of two protected cash-flows with asssociated random gains G_T^1 and G_T^2, a decision in favor of the first one is taken if and only if $RAROC_\alpha[G_T^1] \geq RAROC_\alpha[G_T^2]$ at given confidence levels α. The RAROC criterion tells us that a random gain is preferred to another if its expected value per unit of economic risk capital is greater.

A simple alternative to RAROC is the *inverse coefficient of variation* (ICV) of the random gain defined by

$$ICV[G_T] = \frac{1}{CV[G_T]}, \quad CV[G_T] = \frac{\sqrt{Var[G_T]}}{E[G_T]}, \tag{2.16}$$

which represents the expected value of the random gain per unit of standard deviation. The ICV criterion tells us that G_T^1 is preferred to G_T^2 if and only if $ICV[G_T^1] \geq ICV[G_T^2]$ or equivalently $CV[G_T^1] \leq CV[G_T^2]$.

3. Biatomic Fréchet-Markov return model.

Without further specification of the identically distributed returns $R_1, R_2 ..., R_T$, the ERC and RAROC quantities cannot be calculated. Besides the mean r and standard deviation σ, we will assume that the skewness of the random return R_k equals γ, $k = 1,...,T$. By assuming further biatomic returns, these characteristics uniquely determine R_k.

Proposition 3.1. The support $\{d,u\}, d < u$, and probabilities $\{p, q = 1 - p\}$ of a biatomic random variable with mean r, standard deviation σ and skewness γ are uniquely determined by

$$\begin{aligned} d &= r - \tfrac{1}{2}\sigma\left(\sqrt{4+\gamma^2} - \gamma\right), \\ u &= r + \tfrac{1}{2}\sigma\left(\sqrt{4+\gamma^2} + \gamma\right), \\ p &= \frac{1}{2}\left(1 + \frac{\gamma}{\sqrt{4+\gamma^2}}\right). \end{aligned} \tag{3.1}$$

Proof. The atoms of a standardized biatomic random variable with mean $r = 0$, standard deviation $\sigma = 1$ and skewness γ solve the *algebraic moment problem* of order two, that is the non-linear equations

$$pd + qu = 0,$$
$$pd^2 + qu^2 = 1,$$
$$pd^3 + qu^3 = \gamma.$$

According to Mammana(1954) (see Hürlimann(1998), chapter I, Hürlimann(2002), Appendix II), the atoms d, u are the distinct real zeros of the standard quadratic orthogonal polynomial of degree two $p_2(x) = x^2 - \gamma x - 1$, that is

$$d = \tfrac{1}{2}\left(\gamma - \sqrt{4+\gamma^2}\right), \quad u = \tfrac{1}{2}\left(\gamma + \sqrt{4+\gamma^2}\right).$$

The formulas (3.1) follow immediately. ◊

Let us first calculate the cost of the guarantee (2.4). Following the usual risk-neutral valuation (e.g. Cox and Ross(1976)), the put option price with payoff $(r_g - R_k)_+$ equals

$$\begin{aligned} P(r_g) &= \frac{1}{1+r_f} E^*\left[(r_g - R_k)_+\right] \\ &= \frac{1}{1+r_f} \cdot \left\{(r_g - d)_+ \cdot p^* + (r_g - u)_+ \cdot (1 - p^*)\right\}, \end{aligned} \tag{3.2}$$

where the risk-neutral probability, which satisfies the condition $E^*[R_k] = r_f$, is given by

$$p^* = \frac{u - r_f}{u - d}, \quad d < r_f < u. \tag{3.3}$$

The dependence structure between the returns $R_1, R_2 ..., R_T$ must also be specified. For simplicity, we assume the *Markov chain property*

$$\begin{aligned} &\Pr\left(R_k = x_k \middle| R_j = x_j,\ j = 1,...,k-1\right) = \Pr\left(R_k = x_k \middle| R_{k-1} = x_{k-1}\right), \\ &k = 2,...,T, \end{aligned} \tag{3.4}$$

where the x_k's belong to the supports of the R_k's. It follows that the joint probability function can be calculated from the formula

$$\Pr\left(R_k = x_k, k = 1,...,T\right) = \Pr\left(R_1 = x_1\right) \cdot \prod_{j=2}^{T} \Pr\left(R_j = x_j \middle| R_{j-1} = x_{j-1}\right). \tag{3.5}$$

The conditional probabilities

$$\Pr\left(R_j = x_j \middle| R_{j-1} = x_{j-1}\right) = \frac{\Pr\left(R_{j-1} = x_{j-1}, R_j = x_j\right)}{\Pr\left(R_{j-1} = x_{j-1}\right)} \tag{3.6}$$

depend upon the knowledge of a bivariate distribution for the random couples $\left(R_{j-1}, R_j\right)$, $j = 2,...,T$, which are assumed to be identically distributed. In general, given a random couple (X,Y) with finite discrete support $\{(x_i, y_k), i = 1,...,m, k = 1,...,n\}$, marginals $F_i = \Pr(X \le x_i)$, $G_k = \Pr(Y \le y_k)$, and joint distribution $H_{ik} = \Pr(X \le x_i, Y \le y_k)$, we will assume that the latter follows a *Fréchet copula* such that for some $\theta \in [0,1]$:

$$H_{ik} = (1-\theta)F_i G_k + \theta \min\{F_i, G_k\}. \tag{3.7}$$

This one-parameter positive quadrant dependent statistical model includes independence $(\theta = 0)$ and comonotonocity $(\theta = 1)$. In our situation, the couples $\left(R_{j-1}, R_j\right)$, $j = 2,...,T$, have support $\{(d,d), (d,u), (u,d), (u,u)\}$ and probabilities $\{p_{11}, p_{12}, p_{21}, p_{22}\}$ such that

$$\left(p_{ij}\right) = \begin{pmatrix} p(p+\theta q) & (1-\theta)pq \\ (1-\theta)pq & q(q+\theta p) \end{pmatrix}, \tag{3.8}$$

where $d, u, p, q = 1-p$ have been defined in (3.1), and $\theta \in [0,1]$. The simple model defined by (3.5), (3.6) and (3.8) is called *biatomic Fréchet-Markov return model*. The independent case $\theta = 0$ is called *biatomic random walk* return model while the comonotone case $\theta = 1$ is called *biatomic comonotone* return model.

To evaluate the random variable (2.3), one observes that

$$\begin{aligned} &V_T = \sum_{k=1}^{T} c_k \cdot \prod_{j=k}^{T} Y_k \text{ , or recursively} \\ &V_k = Y_k\left(V_{k-1} + c_k\right), \quad k = 2,...,T, \end{aligned} \tag{3.9}$$

where $Y_k = 1 + r_g + \left(R_k - r_g\right)_+$, $k = 1,...,T$, is a finite sequence of Fréchet-Markov identically distributed biatomic random variables with supports $\{a,b\}, a < b$, and probabilities $\{p, q = 1-p\}$ given by (assume $r_g \ge d = r - \frac{1}{2}\sigma\left(\sqrt{4+\gamma^2} - \gamma\right)$)

$$\begin{aligned} &a = 1 + r_g, \\ &b = 1 + r + \tfrac{1}{2}\sigma\left(\sqrt{4+\gamma^2} + \gamma\right), \\ &p = \frac{1}{2}\left(1 + \frac{\gamma}{\sqrt{4+\gamma^2}}\right). \end{aligned} \tag{3.10}$$

To obtain the $n = 2^T$ atoms of the discrete random variable V_T, consider the set Δ of all zero-one vectors $\delta = (\delta_1,...,\delta_T)$ with $\delta_k \in \{0,1\}$. Then each $\delta \in \Delta$ yields an atom

$$v_\delta = \sum_{k=1}^{T} c_k \cdot \prod_{j=k}^{T} a^{1-\delta_j} b^{\delta_j} \tag{3.11}$$

with probability

$$h_\delta = \Pr(V_T = v_\delta) = p^{1-\delta_1} q^{\delta_1} \cdot \prod_{k=2}^{T} \left\{ (1-\theta) \cdot p^{1-\delta_k} q^{\delta_k} + \theta \cdot \varepsilon(\delta_{k-1}, \delta_k) \right\}, \tag{3.12}$$

where $\varepsilon(x,y)=1$ if $y=x$ and 0 else. To evaluate CVaR of the cash-flow risk $X_T = C_T + L_T - V_T$, reorder and rename the n atoms $x_\delta = C_T + L_T - v_\delta$ of X_T in ascending order such that $x_1 < x_2 < ... < x_n$ and $f_k = \Pr(X_T = x_k), k = 1,...,n$. Using (2.11) one obtains the formula

$$CVaR_\alpha[X_T] = x_{k_\alpha} + \frac{1}{1-\alpha} \cdot \sum_{k=k_\alpha}^{n} (x_k - x_{k_\alpha}) \cdot f_k \,, \tag{3.13}$$

where k_α satisfies the inequality (2.12).

4. <u>Guaranteed annuity-due with constant payments</u>.

The useful special case $c_k = 1, k = 1,...,T$, represents a guaranteed annuity-due with T annual payments of amount 1. The accumulated value at time T of a deterministic annuity-due with constant payments of amount 1 and interest rate j is denoted by

$$S(T,j) = \left(\frac{1+j}{j} \right) \left[(1+j)^T - 1 \right]. \tag{4.1}$$

The protected random annuity with guaranteed interest rate r_g induces the deterministic liability at time T of amount

$$L_T = S(T, r_g) \tag{4.2}$$

and the cost of guarantee (use formula (2.4))

$$\begin{aligned} C_T &= P(r_g) \sum_{t=1}^{T} S(t, r_g)(1+r_f)^{T-t+1} \\ &= P(r_g) \left(\frac{1+r_g}{r_g} \right) \left[(1+r_g)^{T+1} S(T, \frac{r_f - r_g}{1+r_g}) - S(T, r_f) \right]. \end{aligned} \tag{4.3}$$

In general, the mean $\mu[V_T]$ and variance $\sigma^2[V_T]$ of V_T can be calculated numerically using (3.11) and (3.12). For the important biatomic random walk return model, that is the independent case $\theta = 0$, there exists even the explicit expressions (see Burnecki et al.(2001), Corollary 3.1, which corrects some main results in Zaks(2001)) :

$$\mu[V_T]=S(T,r_y),\quad r_Y=E[Y_k]-1=ap+bq-1, \tag{4.4}$$

$$\sigma^2[V_T]=\frac{2(1+r_Y)^{T+1}S(T,g)-(2+r_Y)S(T,f)-(1+r_Y)S(2T,r_Y)+2(1+r_Y)S(T,r_Y)}{r_Y} \tag{4.5}$$

where one sets

$$f=2r_Y+r_Y^2+s_Y^2,\quad s_Y^2=Var[Y_k]=(b-a)^2pq,\quad g=r_Y+\frac{s_Y^2}{1+r_Y}. \tag{4.6}$$

As a general consequence, the inverse coefficient of variation of gain is explicitly given by

$$ICV[G_T]=\frac{\mu[V_T]-L_T-C_T}{\sigma[V_T]}. \tag{4.7}$$

The calculation of CVaR is also rather explicit. First of all, it is convenient to number the zero-one vectors $\delta\in\Delta$ in such a way that $\delta_{k-1}=(\delta_{k-1,1},...,\delta_{k-1,T})$ precedes $\delta_k=(\delta_{k,1},...,\delta_{k,T})$ in the lexicographic order, $k=2,...,n=2^T$. Since $a<b$ one sees immediately that $v_{\delta_{k-1}}<v_{\delta_k}$, $k=2,...,n$. Setting $x_k=C_T+L_T-v_{\delta_{n-k+1}}$ and $f_k=h_{\delta_{n-k+1}}$, $k=1,...,n$, the atoms of X_T form automatically an increasing sequence and the proposed numerical algorithm to evaluate CVaR applies.

A remarkable feature of the put option strategy used to protect the random annuity is the *constant* amount of required *economic risk capital* as long as the loss probability is sufficiently small. In our special situation, this constant is the cost of guarantee. As a consequence, the economic risk capital is always bounded above by the cost of guarantee.

Proposition 4.1. If $\varepsilon<p(p+\theta q)^{T-1}$, $\theta\in[0,1]$, $r_g\geq d=r-\frac{1}{2}\sigma(\sqrt{4+\gamma^2}-\gamma)$, one has

$$CVaR_\alpha[X_T]=VaR_\alpha[X_T]=C_T. \tag{4.8}$$

Proof. For $\delta_1=(0,...,0)$ one has $v_{\delta_1}=S(T,a-1)$ and $h_{\delta_1}=p(p+\theta q)^{T-1}$. Since $\varepsilon<f_n=p(p+\theta q)^{T-1}$ the inequality (2.12) is only satisfied when $k_\alpha=n$, hence

$$CVaR_\alpha[X_T]=VaR_\alpha[X_T]=x_n=C_T+L_T-v_{\delta_1}.$$

The formulas (4.8) follows using (4.2) and (4.3) and the fact that $a=1+r_g$ under the assumption $r_g\geq d$. ◊

The quantity RAROC satisfies a similar property.

Corollary 4.1. If $\varepsilon<p(p+\theta q)^{T-1}$, $\theta\in[0,1]$, $r_g\geq d=r-\frac{1}{2}\sigma(\sqrt{4+\gamma^2}-\gamma)$, one has

$$RAROC_\alpha[G_T]=\frac{\mu[V_T]-S(T,r_g)-C_T}{C_T}. \tag{4.9}$$

Example 4.1.

It is interesting to look at the extreme case of the biatomic comonotone return model obtained when $\theta = 1$. From (3.12) one sees that $h_\delta = 0$ unless $\delta = (0,...,0)$ or $\delta = (1,...,1)$. In the special case of constant payments, one obtains a biatomic random value V_T with support $\{S(T,a-1), S(T,b-1)\}$ and probabilities $\{p, q = 1-p\}$. Then the cash-flow risk X_T is also biatomic with support $\{x_1, x_2\} = \{C_T + L_T - S(T,b-1), C_T + L_T - S(T,a-1)\}$ and probabilities $\{f_1, f_2\} = \{q, p\}$. The weighted average representation of CVaR yields in the biatomic case (Proposition 6 in Rockafellar and Uryasev(2001), formula (2.6) in Hürlimann(2003)):

$$CVaR_\alpha[X_T] = \lambda x_1 + (1-\lambda) x_2, \qquad \lambda = \begin{cases} \frac{q-\alpha}{1-\alpha}, & \varepsilon \ge p, \\ 0, & \varepsilon < p. \end{cases} \tag{4.10}$$

If $\varepsilon < p$ one recovers the assertions of Proposition 4.1 and Corollary 4.1. As a remarkable feature we show that this condition is almost always fulfilled.

Corollary 4.2. If $\theta = 1$, $\varepsilon < \frac{1}{2}$, $\gamma > -\sqrt{\frac{1}{2}\left(\frac{1-2\varepsilon}{\varepsilon}\right)}$, one has $CVaR_\alpha[X_T] = VaR_\alpha[X_T] = C_T$.

Proof. Recall from (3.1) that $p = \frac{1}{2}\left(1 + \frac{\gamma}{\sqrt{4+\gamma^2}}\right)$ with γ the skewness. If $\gamma > 0$ one has $p > \frac{1}{2} > \varepsilon$, hence (4.11) holds. If $\gamma < 0$ the condition $\varepsilon < p$ is equivalent with $\gamma^2 < \frac{1}{2}\left(\frac{1-2\varepsilon}{\varepsilon}\right)$, and is fulfilled by assumption on γ. ◊

In general, from Proposition 4.1 it follows that there is a maximum time horizon for which CVaR=VaR is maximum. For the confidence $\alpha = 0.99$, Table 4.1 lists the maximum time

$$T_{\max} = 1 + \left[\frac{\ln(\varepsilon) - \ln(p)}{\ln(p + \theta q)}\right], \quad \varepsilon = 1 - \alpha, \tag{4.11}$$

in function of the skewness γ and the dependence parameter θ.

Table 4.1 : maximum time horizon for maximum CVaR

γ \ θ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	6	7	8	10	11	14	18	25	38	77
0.25	7	9	10	12	14	17	21	29	44	90
0.5	9	10	12	14	17	20	26	35	53	107
0.75	11	13	15	17	20	24	31	42	63	128
1.0	14	15	18	20	24	29	37	50	76	153

Since ERC is bounded above by the cost of guarantee, it is interesting to analyze when this quantity is maximal. This happens in case the put option price $P(r_g)$ is maximal. We determine the maximum put option price as function of the skewness parameter γ. In general, replacing $P(r_g)$ by its maximal value yields the maximum cost of guarantee under the biatomic Fréchet-Markov return model. Since CVaR is translation-invariant (property of a coherent risk measure), one has $CVaR_\alpha[X_T] = C_T + L_T - CVaR_\alpha[-V_T]$. Therefore, using the maximum cost of guarantee, CVaR is replaced by a prudent upper bound.

Proposition 4.2. Suppose that $r_g < r_f < r$, $(r-r_f)(r-r_g) < \sigma^2$, and that the returns $R_k, k = 1,...,T$ form an identically distributed sequence of biatomic random variables with mean r, standard deviation σ, and arbitrary skewness γ. Then the put option price (3.2) with payoff $(r_g - R_k)_+$, given by

$$P(r_g) = -\frac{1}{2}(r_f - r_g) + \frac{\sigma^2 - (r-r_f)(r-r_g)}{\sigma\sqrt{4+\gamma^2}} - \frac{1}{2}\frac{(2r - r_f - r_g)}{\sqrt{4+\gamma^2}}, \quad d \le r_g \le u, \tag{4.12}$$

is maximal at the skewness parameter

$$\gamma_0 = -2\sigma \cdot \frac{(r-r_f)+(r-r_g)}{-(r-r_f)(r-r_g)+\sigma^2} < 0. \tag{4.13}$$

Proof. Under the restriction $d \le r_g \le u$ one finds using (3.1)-(3.3) the expression (4.12) for the put option price. View this as a function $f(\gamma)$ of the skewness parameter. One has

$$f'(\gamma) = -\frac{A\gamma + 4B}{(4+\gamma^2)^{\frac{3}{2}}}, \quad A = \frac{\sigma^2 - (r-r_f)(r-r_g)}{\sigma}, \quad B = \frac{1}{2}(2r - r_f - r_g).$$

The stationary point $\gamma_0 = -\frac{4B}{A}$, which is equal to (4.13), yields a maximum because $f''(\gamma_0) = -\frac{A}{8(1+4\frac{B^2}{A^2})^{\frac{3}{2}}} < 0$. With the maximizing value $\gamma = \gamma_0$, it is not difficult to show that the condition $d \le r_g \le u$ is fulfilled provided $(r-r_f)(r-r_g) < \sigma^2$. The result is shown. ◊

Table 4.2 illustrates our findings at the specific biatomic random walk return model with $\theta = 0$ and parameters $r_g = 4.25\% < r_f = 5\% < r = 5.81\%$, $\sigma = 1.9558\%$, $\gamma = 0.3032$. Similar calculations could be done for other values $\theta > 0$ of the dependence parameter. These values of r, σ, γ, which are borrowed from Das(2002), Table 1, p.29, are the summary statistics for the Fed Funds rate over the period January 1988 to December 1997. The specification (3.10) of the random returns Y_k, required for the evaluation of the random annuity value V_T, is done using these values. The "guaranteed" biatomic return $Y_k - 1$ jumps between the values $a - 1 = 4.25\%$ and $b - 1 = 8.085\%$. With the observed skewness the return R_k jumps between 4.128% and 8.085%. Compare this with the observed range, which variies between the minimum rate 2.58% and the maximum rate 10.71%. With the skewness $\gamma_0 = -3.61909$, which maximizes the cost of guarantee after Proposition 4.3, the obtained range between -1.773% and 6.314% is certainly too much negatively skewed. A compromise for the calculation of $P(r_g)$ is to take $\gamma = -1.046$ for which R_k jumps between 2.58% and 6.994%. This skewness value is used for a prudent estimation of the cost of guarantee in Table 4.2. For $T > 8$ there is a considerable decrease in ERC per unit of liability. This effect is a consequence of the chosen confidence level.

Table 4.2 : ERC and RAROC of a guaranteed annuity-due

T	L_T	$\frac{100 \cdot C_T}{L_T}$	$\frac{100 \cdot CVaR}{L_T}$	$\mu[V_T]$	$\sigma[V_T]$	$ICV[G_T]$	RAROC
1	1.043	0.75	0.75	1.059	0.019	0.445	1.072
2	2.129	1.14	1.14	2.180	0.044	0.598	1.078
3	3.262	1.54	1.54	3.367	0.076	0.714	1.084
4	4.443	1.94	1.94	4.624	0.116	0.813	1.090
5	5.675	2.35	2.35	5.954	0.162	0.901	1.096
6	6.958	2.77	2.77	7.363	0.217	0.978	1.103
7	8.297	3.19	3.19	8.855	0.279	1.053	1.109
8	9.692	3.62	3.62	10.434	0.349	1.121	1.115
9	11.146	4.06	3.91	12.107	0.429	1.184	1.165
10	12.662	4.51	4.10	13.877	0.518	1.244	1.241

5. Triatomic Fréchet-Markov return model.

Suppose that the identically distributed returns $R_k, k = 1,...,T$, have the mean r, standard deviation σ, skewness γ and kurtosis γ_2. If one assumes triatomic returns, these characteristics define the following one-parameter family of returns R_k.

Proposition 5.1. Suppose that $\Delta = \gamma_2 - \gamma^2 + 2 \geq 0$. The support $\{x_1, x_2, x_3\}$, $x_1 < x_2 < x_3$, and probabilities $\{p_1, p_2, p_3\}$ of a triatomic random variable with mean r, standard deviation σ, skewness γ and kurtosis γ_2, are determined as follows :

$$\begin{aligned} x_1 &= r + \sigma x \in (-\infty, r + \sigma c], \quad c = \tfrac{1}{2}\left(\gamma - \sqrt{4+\gamma^2}\right), \\ x_2 &= r + \sigma\varphi(x, \psi(x)) \in [r + \sigma c, r + \sigma\bar{c}], \quad \bar{c} = \tfrac{1}{2}\left(\gamma + \sqrt{4+\gamma^2}\right), \\ x_3 &= r + \sigma\psi(x) \in [r + \sigma\bar{c}, \infty), \end{aligned} \tag{5.1}$$

$$p_i = p\left(\frac{x_i - r}{\sigma}\right), \quad i = 1,2,3, \tag{5.2}$$

where the functions $\varphi(u,v)$, $\psi(u)$ and $p(u)$ are defined by

$$\varphi(u,v) = \frac{\gamma - u - v}{1 + uv}, \tag{5.3}$$

$$\psi(u) = \frac{1}{2}\left(\frac{C(u) - \sqrt{C(u)^2 + 4q(u)D(u)}}{q(u)} \right),$$
$$C(u) = \gamma q(u) + \Delta u, \quad D(u) = \Delta + q(u), \tag{5.4}$$
$$q(u) = 1 + \gamma u - u^2, \quad \Delta = 2 + \gamma_2 - \gamma^2,$$
$$p(u) = \frac{\Delta}{q(u)^2 + \Delta(1+u^2)}. \tag{5.5}$$

<u>Proof.</u> It suffices to look at standardized triatomic random variables with atoms $z_i = \frac{x_i - r}{\sigma}$, $i = 1,2,3$. These atoms solve the algebraic moment problem of order three. By Mammana(1954) (see Hürlimann(1998), chapter I, Hürlimann(2002), Appendix II) they are the distinct real zeros of the standard cubic orthogonal polynomial of degree three $p_3(x)$, which satisfies the three linear expected value equations

$$E\left[Z^i p_3(Z)\right] = 0, \quad i = 0,1,2. \tag{5.6}$$

The condition

$$E\left[p_3(Z)\right] = E\left[(Z - z_1)(Z - z_2)(Z - z_3)\right] = \gamma - (z_1 + z_2 + z_3) - z_1 z_2 z_3 = 0$$

implies the relationship

$$z_2 = \varphi(z_1, z_3). \tag{5.7}$$

Inserted into the condition

$$E\left[Zp_3(Z)\right] = E\left[Z(Z - z_1)(Z - z_2)(Z - z_3)\right] = \gamma_2 + 3 - (z_1 + z_2 + z_3)\gamma + (z_1 z_2 + z_1 z_3 + z_2 z_3) = 0,$$

one obtains that z_3 is solution of the quadratic equation

$$q(z_1) z_3^2 - C(z_1) z_3 - D(z_1) = 0, \tag{5.8}$$

hence $z_3 = \psi(z_1)$ as defined in (5.4). The probabilities take the values

$$p_1 = \frac{1 + z_2 z_3}{(z_2 - z_1)(z_3 - z_1)}, \quad p_2 = \frac{-(1 + z_1 z_3)}{(z_2 - z_1)(z_3 - z_2)}, \quad p_3 = \frac{1 + z_1 z_2}{(z_3 - z_1)(z_3 - z_2)}. \tag{5.9}$$

Since $z_1 < z_2 < z_3$ one must have $1 + z_2 z_3 \geq 0$, $1 + z_1 z_3 \leq 0$, $1 + z_1 z_2 \geq 0$. Since $z_1 z_3 \leq -1$ one must have $z_1 < 0 < z_3$, hence also $\bar{z}_3 < 0 < \bar{z}_1$, where $\bar{x} = -x^{-1}$ for $x \neq 0$. It follows that

$$\bar{z}_3 \cdot (1 + z_2 z_3) = \bar{z}_3 - z_2 \leq 0,$$
$$\bar{z}_3 \cdot (1 + z_1 z_3) = \bar{z}_3 - z_1 \geq 0,$$
$$\bar{z}_1 \cdot (1 + z_1 z_2) = \bar{z}_1 - z_2 \geq 0,$$

which implies the inequalities

$$z_1 \le \bar{z}_3 < 0, \quad \bar{z}_3 \le z_2 \le \bar{z}_1 . \tag{5.10}$$

Since $z_2 = \varphi(z_1, z_3)$ the second inequalities in (5.10) are equivalent with

$$z_3^2 - \gamma z_3 - 1 \ge 0, \quad z_1^2 - \gamma z_1 - 1 \ge 0 . \tag{5.11}$$

Since $z_1 < 0 < z_3$ one must have $(z_1, z_3) \in (-\infty, c] x [\bar{c}, \infty)$ and $z_2 \in [c, \bar{c}]$. This shows (5.1). To obtain the representation (5.2), note that $z_2 = \varphi(z_1, \psi(z_1))$ and $z_3 = \psi(z_1)$ are solutions of the quadratic equation $q(z_1)z^2 - C(z_1)z - D(z_1) = 0$. One calculates

$$(z_2 - z_1)(z_3 - z_1) = z_1^2 - (z_2 + z_3)z_1 + z_2 z_3 = z_1^2 - \frac{C(z_1)}{q(z_1)} z_1 - \frac{D(z_1)}{q(z_1)} = -\frac{q(z_1)^2 + \Delta(1 + z_1^2)}{q(z_1)},$$

$$1 + z_2 z_3 = -\frac{\Delta}{q(z_1)}.$$

Inserted into (5.9) one gets $p_1 = p(z_1)$. Making cyclic permutations of z_1, z_2, z_3 one obtains $p_i = p(z_i), \quad i = 2,3$. ◊

To perform our evaluations, a specification of R_k using Proposition 5.1 must be made. We suggest to choose R_k such that a return less than $x_1 = r + \sigma x$ occurs only with the loss probability $\varepsilon = 1 - \alpha$, where α is the confidence level used to calculate CVaR. Then x is uniquely determined by the condition

$$p(x) = \varepsilon, \quad x \in (-\infty, c]. \tag{5.12}$$

The cost of guarantee (2.4) is in great part determined by the put option price $P(r_g) = \frac{1}{1 + r_f} E^*\left[(r_g - R_k)_+\right]$. To fulfill the risk-neutral probability condition $E^*[R_k] = r_f$, we suggest to replace the probabilities p_i by risk-neutral probabilities $p_i^* = p\left(\frac{x_i^* - r}{\sigma}\right)$, with $x_1^* = r + \sigma x^*$, $x_2^* = r + \sigma\varphi(x^*, \psi(x^*))$, $x_3^* = r + \sigma\psi(x^*)$. Explicitly, x^* solves the equation

$$[r + \sigma x]p(x^*) + [r + \sigma\varphi(x, \psi(x))]p(\varphi(x^*, \psi(x^*))) + [r + \sigma\psi(x)]p(\psi(x^*)) = r_f . \tag{5.13}$$

The put option price is then given by

$$P(r_g) = \frac{1}{1 + r_f} \cdot \begin{Bmatrix} (r_g - r - \sigma x)_+ p(x^*) + (r_g - r - \sigma\varphi(x, \psi(x)))_+ p(\varphi(x^*, \psi(x^*))) \\ + (r_g - r - \sigma\psi(x))_+ p(\psi(x^*)) \end{Bmatrix} . \tag{5.14}$$

Assuming the Markov chain property (3.4), it remains to specify the dependence structure between the random couples (R_{j-1}, R_j), $j = 2,...,T$, which are again assumed to be identically distributed. As in Section 3, these couples follow Fréchet copulas and have support $\{(x_i, x_j), 1 \le i, j \le 3\}$ and probabilities $\{p_{ij}, 1 \le i, j \le 3\}$ such that

$$p_{ij} = \begin{cases} p_i[p_i + \theta(1-p_i)], & j = i, \\ (1-\theta) p_i p_j, & j \neq i, \end{cases} \tag{5.15}$$

where the x_i's and p_i's have been defined in Proposition 5.1, and $\theta \in [0,1]$. This simple model is called *triatomic Fréchet-Markov return model*. The independent case $\theta = 0$ is called *triatomic random walk* return model while the comonotone case $\theta = 1$ is called *triatomic comonotone* return model.

To evaluate (2.3) we use the representation (3.4), where $Y_k, k = 1,...,T$, is a finite sequence of Fréchet-Markov identically distributed triatomic random variables with supports $\{a_1, a_2, a_3\}$, $a_1 < a_2 < a_3$, and probabilities $\{p_1, p_2, p_3\}$ given by (assume $x_1 \le r_g \le x_2 < x_3$)

$$\begin{aligned} &a_1 = 1 + r_g, \quad p_1 = p(x), \\ &a_2 = 1 + r + \sigma\varphi(x, \psi(x)), \quad p_1 = p(\varphi(x, \psi(x))), \\ &a_3 = 1 + r + \sigma\psi(x), \quad p_3 = p(\psi(x)). \end{aligned} \tag{5.16}$$

Note that if $x_2 < r_g$ then Y_k is a biatomic random variable, for which the results of Section 3 applies. To obtain the $n = 3^T$ atoms of the discrete random variable V_T , consider the set Δ of all vectors $\delta = (\delta_1,...,\delta_T)$ with $\delta_k \in \{-1,0,1\}$. Then each $\delta \in \Delta$ yields an atom

$$v_\delta = \sum_{k=1}^{T} c_k \cdot \prod_{j=k}^{T} a_1^{\frac{1}{2}(|\delta_j| - \delta_j)} a_2^{1-|\delta_j|} a_3^{\frac{1}{2}(|\delta_j| + \delta_j)} \tag{5.17}$$

with probability

$$\begin{aligned} h_\delta = \Pr(V_T = v_\delta) = &\, p_1^{\frac{1}{2}(|\delta_1| - \delta_1)} p_2^{1-|\delta_1|} p_3^{\frac{1}{2}(|\delta_1| + \delta_1)} \\ &\cdot \prod_{k=2}^{T} \left\{ (1-\theta) \cdot p_1^{\frac{1}{2}(|\delta_k| - \delta_k)} p_2^{1-|\delta_k|} p_3^{\frac{1}{2}(|\delta_k| + \delta_k)} + \theta \cdot \varepsilon(\delta_{k-1}, \delta_k) \right\} \end{aligned} \tag{5.18}$$

where $\varepsilon(x, y) = 1$ if $y = x$ and 0 else. Then proceed as at the end of Section 3.

Let us illustrate the obtained results at the guaranteed annuity-due with constant payments considered in Section 4 under the triatomic random walk return model with $\theta = 0$. As numerical example we use the data of Section 4 completed with the kurtosis parameter $\gamma_2 = -0.8304$. The triatomic return has support $\{x_1 = -0.328\%, x_2 = 4.493\%, x_3 = 8.395\%\}$. Its span $x_3 - x_1 = 8.723\%$ is comparable to the observed span $10.71\% - 2.58\% = 8.13\%$. The triatomic returns $Y_k - 1$ have the support $\{a_1, a_2, a_3\} = \{4.25\%, 4.493\%, 8.395\%\}$ and the probabilities $\{p_1, p_2, p_3\} = \{0.01, 0.64011, 0.34989\}$. The cost of guarantee per unit of liability lies between 19 and 94 basic points below the prudent estimation of Section 4, and appears to us as a more "realistic" estimation. A dramatic improvement of the triatomic return model is felt in ERC and RAROC calculations. At the confidence level $\alpha = 0.99$ much less ERC is required. Our calculations are summarized in Table 5.1.

Table 5.1 : ERC and RAROC of a guaranteed annuity-due

T	L_T	$\frac{100 \cdot C_T}{L_T}$	$\frac{100 \cdot CVaR}{L_T}$	$\mu[V_T]$	$\sigma[V_T]$	$ICV[G_T]$	RAROC
1	1.043	0.56	0.56	1.059	0.019	0.549	1.755
2	2.129	0.85	0.69	2.179	0.043	0.737	2.167
3	3.262	1.14	0.85	3.365	0.075	0.880	2.384
4	4.443	1.44	1.00	4.621	0.114	1.001	2.554
5	5.675	1.74	1.15	5.950	0.159	1.107	2.706
6	6.958	2.05	1.30	7.357	0.213	1.203	2.827
7	8.297	2.36	1.46	8.846	0.274	1.291	2.914
8	9.692	2.68	1.63	10.423	0.343	1.373	2.977

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