Diversification

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Abstract

In this paper we analyze the diversification benefit. We draw attention to the model and parameter risks which may have a very important influence on the calculation of the required solvency level of a financial conglomerate. We also comment on value creation for a financial conglomerate and show that it may be justified to write business with negative margins. The very simple numerical applications used in the paper exemplify why discussions between financial conglomerates and regulators should be very detailed in order to achieve a transparent compromise on the internal models that insurance companies will develop in the future to assess their solvency.

Keywords

Required solvency level, Value at Risk, Tail Value at Risk, Risk adjusted capital, diversification benefit, value creation, stochastic dependencies, model risk, regulator, internal models.

Diversification

1 Introduction

Insurance business is characterized by the inversion of the business cycle. Insurers receive premiums first and pay losses later. Therefore insurers can give financial discounts to the premiums they charge to policyholders. It also implies that insurers must hold an appropriate amount of capital in order to minimize the chance that they become insolvent. Indeed, policyholders do not want to exchange the risks they send to the insurance carrier with default risk.

This paper is devoted to the calculation of the required solvency level. In particular it tackles the diversification benefit.

An insurance company faces a large number of risks. According to the International Actuarial Association (2004), risks may be classified as follows :

- Underwriting risk : this is the risk that there will be a deviation between actual and forecast losses. This may happen because of statistical fluctuations or because the models or parameters used are wrong. A fraction of this risk is typically transferred from insurers to reinsurers through reinsurance agreements. A particular example of underwriting risk, that is usually ceded to reinsurers is the "extreme events" risk, i.e. the risk of high-impact and low frequency events.
- Credit risk : this is the risk of default and change in credit quality of issuers of securities, counter-parties and intermediaries, to whom the company has an exposure. Reinsurers are typically a particular counter-party for insurance companies.
- Market risk : market risk arises from the volatility of market prices of assets. Market risk involves the exposure to movements in the level of financial variables such as stock prices, interest rates, exchange rates or commodity prices.
- Operational risk : operational risk is the risk of loss resulting from inadequate or failed internal processes, people, systems or from external events.

Actuaries typically concentrate on the underwriting risk. It is however clear that the other risks may be of predominant importance as well. In this paper we will concentrate on underwriting risk and we will make some remarks about the possible stochastic dependence between underwriting risk and other risks.

For a given type of business, policies have a certain volatility : this is the stochastic deviation of the result around the mean. When risks are independent, the coefficient of variation of a portfolio decreases with the size of the portfolio. Therefore insurers tend to write large portfolios of independent risks in order to decrease the relative volatility of their business and as a consequence their economic capital. This is why actuaries usually claim that the underwriting risk is diversifiable.

When it comes to the measure of the risk, it is another story. Indeed, the actuary does not know the model which is behind the loss generating process. In fact the measure of the risks is characterized by volatility and uncertainty.

Uncertainty is the risk that the models used to estimate the claims or other relevant processes are misspecified or that the parameters within the models are misestimated. In most situations the actuary is not in a position that he can translate reality in an exact actuarial model. There always remains a chance that the model is wrong. Uncertainty risk is obviously not diversifiable for the insurance company. Indeed, if the model used is wrong, the measure of the risk of all the policies will be affected by the error. One may argue that the insurer can diversify the model risk across the lines of business he is writing. Unfortunately the number of lines of business being quite small, it will not be possible to reach the kind of diversification obtained for the volatility risk on thousands of policies. Some examples of model and parameter risk are :

- future inflation is badly estimated (all policies of all lines of business will be influenced: in this case diversification accross the lines of business is even not possible)
- longevity risk : this is particularly important as it will take a very long time before effectively knowing whether the models used are wrong or not
- using a light-tailed distribution where the distribution is in reality heavy-tailed
- not recognizing that the speed of payment is changing compared to past observations
- not recognizing dependencies between lines of business or policies
- estimated parameters of a distribution may not be correct due to the low size of the sample.

It may be possible for an insurance company to hedge some of the model risks. Longevity risk and mortality risk can compensate each other as soon as the exposures are comparable.

When we analyze the situation of the shareholders of the insurance company we may argue that the uncertainty risk is diversifiable. Since they hold a sufficient number of stocks in their portfolio, they are able to diversify the uncertainty risk. Nevertheless the regulator will demand an amount of economic capital which is based on the situation of the insurance company because its aim is to protect the policyholders against failure of the insurance company.

The rest of the paper is organized as follows. In Section 2 we use a basic numerical example in order to introduce the diversification benefit. Model risk is introduced in Section 3 where we further analyze the diversification benefit in presence of model risk. Section 4 insists on the subadditivity property for risk measures and rules the Value at Risk out. In Section 5 we explicitly introduce dependencies in the modelling of losses. In section 6 we quantify the lack of diversification due to mixture models arising from parameter risk. A multivariate normal model is proposed in Section 7 in order to analyze the solvency of an insurance portfolio. The multivariate normal model is extended in Section 8. We conclude in Section 9 and provide recommendations about the discussions that should take place between insurance companies and regulators in order to set up transparent rules about the future internal models that will be used by insurers and approved by regulators.

2 Diversification Benefit

Assume a fire portfolio with n risks, all with an insured sum equal to 1, which can only generate total claims. Assume that in a given year, the policyholders may claim at most once. The probability of making a claim is p. We assume that the risks are independent.

In mathematical terms, we can propose the following model : let X_i be the loss of policy i, $1 \leq i \leq n$. Then $\mathbb{P}[X_i = 1] = p$ and $\mathbb{P}[X_i = 0] = 1 - p$, $1 \leq i \leq n$. We assume that the $X_1, X_2, \ldots, X_n$ are stochastically independent.

The required solvency level (RSL) of an insurance company is the amount that the insurer has to hold in order to be able to compensate the policyholders with a given level of security. The RSL should be determined by the regulator. If we substract the amount of money that is borrowed from the policyholders, i.e. the premiums, we obtain the capital, or risk adjusted capital, provided by the investors.

Let S be the annual aggregate claim amount with cumulative density function $F_S(x) = \mathbb{P}[S \leq x]$. We will first assume that the RSL is given by the Value at Risk of level ϵ :

$$\mathrm{VaR}_\epsilon(S) = \inf\{x \in \mathbb{R} | F_S(x) \geq \epsilon\}\,,\ 0 < \epsilon < 1.$$

The required solvency level of an insurer writing n risks can easily be calculated. Indeed, $S = X_1 + \cdots + X_n$ is binomially distributed with parameters n and $p = 1/10\,000$:

$$\begin{aligned} S &\sim \mathrm{Bin}(n,p) \\ \mathbb{P}[S = s] &= \frac{n!}{s!(n-s)!} p^s (1-p)^{n-s}\,,\ s = 0, 1, \ldots, n. \end{aligned}$$

In table 2.1, we give the Value at Risk at different security levels (ϵ) for our portfolio :

n/ϵ	95%	99%	99.9%	99.99%	99.999%
50	0	0	1	1	2
500	0	1	2	2	3
5 000	2	3	4	5	6
50 000	9	11	13	15	17

Table 2.1: VaR in function of the size of the portfolio and the security level with $p = 1/10\,000$

We immediately observe the diversification benefit when writing independent policies. When the size of the portfolio is multiplied by 10, the required solvency level is increased by a factor much less than 10. This is because a large portfolio is less volatile than a small portfolio according to the law of large numbers.
Note that when the portfolio is large enough, the diversification benefit does not play anymore. This is because the standard deviation becomes so small compared to the mean that the random variable is almost degenerated in its mean. Therefore the Value at Risk is almost equal to its mean. This implies that for very large portfolios when the size of the portfolio further increases, the Value at Risk must increase at the same speed. Let us illustrate this effect with the following numerical example. Assume that the X_i's are independent and identically distributed following a normal distribution with parameters $\mu = 1$ and $\sigma = 3$:

$$\begin{aligned} X_i &\sim \text{Nor}(\mu = 1, \sigma = 3) \\ f_{X_i}(x) &= \frac{1}{\sigma\sqrt{2\pi}} \exp(-(x-\mu)^2/2\sigma^2). \end{aligned}$$

Then it is well-known that

$$S \sim \text{Nor}(n\mu, \sqrt{n}\sigma).$$

Table 2.2 gives the Value at Risk at the level $\epsilon = 99\%$, the relative increase of the Value at Risk and the ratio between the Value at Risk and the mean for different sizes of the porfolio:

n	$VaR_{99\%}(S)$	Relative increase	$\frac{VaR_{99\%}(S)}{\mathbb{E}S}$
1	7.98		7.98
2	11.87	1.49	5.93
4	17.96	1.51	4.49
8	27.74	1.54	3.47
16	43.92	1.58	2.74
32	71.48	1.63	2.23
64	119.83	1.68	1.87
128	206.96	1.73	1.62
256	367.66	1.78	1.44
512	669.92	1.82	1.31
1 024	1 247.33	1.86	1.22
2 048	2 363.83	1.90	1.15
4 096	4 542.66	1.92	1.11
8 192	8 823.67	1.94	1.08
16 384	17 277.32	1.96	1.05
32 768	34 031.34	1.97	1.04
65 536	67 322.63	1.98	1.03
131 072	133 598.68	1.98	1.02
262 144	265 717.26	1.99	1.01
524 288	529 341.35	1.99	1.01
1 048 576	1 055 722.52	1.99	1.01
2 097 152	2 107 258.71	2.00	1.00

Table 2.2: Value at Risk, its relative increase and ratio $\mathrm{VaR}_{99\%}(S)/\mathbb{E}S$ for different sizes (n) of the portfolio

We also observe in table 2.1 that the increase in required solvency level in function of the security level is less pronounced for larger portfolios. This is due to the fact that the coefficient of variation and the skewness of the portfolio decrease with the size of the portfolio:

n	50	500	5 000	50 000
$\frac{\sigma}{\mu}$	14.14	4.47	1.41	0.45
γ	14.14	4.47	1.41	0.45

Table 2.3: Coefficient of variation (σ/μ) and skewness (γ) in function of the size of the portfolio

For a binomial distribution, we have

$$\begin{aligned} \frac{\sigma}{\mu} &= \frac{1-p}{\sqrt{np(1-p)}} \\ \gamma &= \frac{1-2p}{\sqrt{np(1-p)}}. \end{aligned}$$

3 Model Risk

Now assume that there is uncertainty about the model we are using. A first type of model risk is the uncertainty about claim probability. Assume that there is 1% chance that the claim probability is 1/1 000 and 99% chance that it is 1/11 000. On average this gives a probability of making a claim equal to 1/10 000.

Now the distribution of the aggregate claims is a mixture of binomially distributed random variables :

$$S|\Theta \sim \text{Bin}(n, \Theta)$$

where $\mathbb{P}[\Theta = 1/1\,000] = 0.01$ and $\mathbb{P}[\Theta = 1/11\,000] = 0.99$. We have $\mathbb{E}\Theta = 1/10\,000$.
In table 3.1, the Value at Risk for this model is summarized :

n/ϵ	95%	99%	99.9%	99.99%	99.999%
50	0	0	1	1	2
500	0	1	2	3	4
5 000	2	3	8	11	13
50 000	9	20	59	67	73

Table 3.1: VaR in function of the size of the portfolio and the security level with model risk of type 1.

Observe that the required solvency level is higher due to the danger implied by the alternative model for claims. Observe also that the larger the portfolio, the larger the difference in the required solvency level between the cases with and without model risk. This represents the non-diversifiable effect of the model risk. To illustrate this better, let us analyze the situation when n further increases. Table 3.2 gives the evolution of the RSL for $\epsilon = 99.999\%$ when n becomes very large. We make the assumption that $S|\Theta \sim \text{Nor}(n\Theta, \sqrt{n\Theta(1-\Theta)})$.

	VaR(S)			VaR(S)/$\mathbb{E}S$	
n	Without model risk	With model risk	Ratio	Without model risk	With model risk
50 000	14.53	71.84	4.94	2.91	14.37
100 000	23.49	130.89	5.57	2.35	13.09
200 000	39.07	243.68	6.24	1.95	12.18
400 000	66.97	461.77	6.89	1.67	11.54
800 000	118.15	887.36	7.51	1.48	11.09
1 600 000	213.95	1 723.55	8.06	1.34	10.77
3 200 000	396.29	3 374.72	8.52	1.24	10.55
6 400 000	747.89	6 647.10	8.89	1.17	10.39
12 800 000	1 432.58	13 149.45	9.18	1.12	10.27
25 600 000	2 775.79	26 094.19	9.40	1.08	10.19

Table 3.2: $\text{VaR}_{99.999\%}(S)$ in function of the size of the portfolio with and without model risk of type 1.

Note that approximating binomial distributions by a normal distribution will not provide excellent results in extreme tails, as is the case here ($\epsilon = 99.999\%$). When we compare the results of the first row of table 3.2 (14.53 and 71.84) with the bottom right figures of table 2.1 and 3.1 (17 and 73), we observe that there is some relative error with the normal approximation.
We observe in table 3.2 that our RSL increases much faster with model risk than without model risk. In fact when the size of the portfolio increases, $\frac{\text{VaR}(S)}{\mathbb{E}S}$ tends to 1. On the other hand with model risk we observe that economic capital is needed even for very large portfolios. This is due to the lack of diversification effect.

A second type of model risk could be a perturbation of all the risks with an identical random effect. Let us denote the random effect by V. We then have the following outcomes of the model : $X_i(1+V)$ where we assume that the $X_1, X_2, \ldots, X_n$ are independent of V. The aggregate claims immediately writes

$$T = X_1(1+V) + \cdots + X_n(1+V) = (1+V)S$$

where $S \sim \text{Bin}(n,p)$ with $p = 1/10\,000$.

Assume the following distribution for the random effect V :

$$\begin{aligned} \mathbb{P}[V=-0.2] &= 0.25 \\ \mathbb{P}[V=-0.1] &= 0.50 \\ \mathbb{P}[V=0] &= 0.15 \\ \mathbb{P}[V=1] &= 0.10 \end{aligned}$$

such that $\mathbb{E}V = 0$.
The distribution function of T is given by

$$\mathbb{P}[T \leq s] = \sum_v \mathbb{P}[V = v]\mathbb{P}[S \leq s/(1+v)].$$

In table 3.3, we summarize the Value at Risk at level ϵ for this model :

n/ϵ	95%	99%	99.9%	99.99%	99.999%
50	0	0	1	2	2
500	0	1	2	4	4
5 000	2	4	6	8	10
50 000	10	16	22	26	30

Table 3.3: VaR_ϵ in function of the size of the portfolio and the security level with model risk of type 2.

We can make the same remarks as the ones we made about model risk of type 1.

4 Subadditivity

It is now well known that the value at risk is not a subadditive risk measure. In other words, it might be the case that the VaR of a conglomerate is not lower than the sum of the VaR of the components of the conglomerate, i.e. the VaR would not recognize the diversification benefit. Let us take an example in order to illustrate this :

Scenario	Probability	U_1	U_2	$U_1 + U_2$
1	80%	0	0	0
2	10%	0	1	1
3	10%	1	0	1

Table 4.1: VaR is not subadditive.

From table 4.1, we have

$$\begin{aligned} \text{VaR}_{85\%}(U_1) &= 0 = \text{VaR}_{85\%}(U_2) \\ \text{VaR}_{85\%}(U_1 + U_2) &= 1 \end{aligned}$$

leading to

$$\text{VaR}_{85\%}(U_1 + U_2) > \text{VaR}_{85\%}(U_1) + \text{VaR}_{85\%}(U_2).$$

Furthermore the VaR does not tell how bad is bad. It does not give an idea about the severity of the default. These are the reasons why other measures of risk have been introduced. In this paper, we use a risk measure that is related to the VaR, namely the Tail Value at Risk, which is a subadditive measure, i.e. respecting

$$\rho(U_1 + U_2) \leq \rho(U_1) + \rho(U_2)$$

where ρ denotes the chosen risk measure.

The Tail Value at Risk at level ϵ is the average of the quantiles above the Value at Risk at the level ϵ :

$$\text{TVaR}_\epsilon(U) = \frac{1}{1-\epsilon}\int_\epsilon^1 \text{VaR}_q(U)dq\,,\; 0<\epsilon<1.$$

For continuous random variables, we have that the Tail Value at Risk is equal to the Conditional Tail Expectation :

$$\text{CTE}_\epsilon(U) = \mathbb{E}[U|U > \text{VaR}_\epsilon(U)]\,,\; 0<\epsilon<1.$$

For continuous random variables, the conditional tail expectation represents the expectation of the top $(1-\epsilon)\%$ losses.

For normally distributed risks with mean μ and standard deviation σ ($U \sim \text{Nor}(\mu,\sigma)$), it is well known (see e.g. Panjer (2001)) that

$$\begin{aligned} \text{VaR}_\epsilon(U) &= \mu + \sigma\Phi^{-1}(\epsilon) \\ \text{CTE}_\epsilon(U) = \text{TVaR}_\epsilon(U) &= \mu + \sigma\frac{\phi(\Phi^{-1}(\epsilon))}{1-\epsilon} \end{aligned}$$

where ϕ (resp. Φ) denotes the density function (resp. the cumulative distribution function) of a standard normal random variable ($\text{Nor}(0,1)$).

Artzner et al. (1999) make some recommendations about the set of axioms that should be satisfied by risk measures. Tail Value at Risk is a coherent risk measure in the sense of Artzner et al. (1999). As from now we will compute the Tail Value at Risk in order to get our required solvency level. More details and proofs may be found e.g. in Denuit et al. (2005). See also Venter (2004) for a discussion on capital allocation.

5 Introducing Dependencies

The model proposed in order to analyze the model risk of type 1 is a model where we have introduced dependencies. Indeed we have that

$$\begin{aligned} \mathbb{P}[X_i = 1|\Theta] &= \Theta \quad , \quad i = 1,\ldots,n \\ \mathbb{P}[X_i = 0|\Theta] &= 1-\Theta \quad , \quad i = 1,\ldots,n \end{aligned}$$

with $X_1, X_2, \ldots, X_n$ being conditionally independent given Θ, and

$$\begin{aligned} \mathbb{P}[\Theta = 1/1\,000] &= 0.01 \\ \mathbb{P}[\Theta = 1/11\,000] &= 0.99 \end{aligned}$$

We therefore have introduced dependency between the unconditional $X_i's$. We have

$$\begin{aligned}
\mathbb{E}X &= \mathbb{E}[\mathbb{E}[X|\Theta]] \\
&= \mathbb{E}[\Theta] \\
&= 1/10000 \\
\mathbb{V}arX &= \mathbb{E}[\mathbb{V}ar[X|\Theta]] + \mathbb{V}ar[\mathbb{E}[X|\Theta]] \\
&= \mathbb{E}[\Theta(1-\Theta)] + \mathbb{V}ar[\Theta] \\
&= \mathbb{E}[\Theta](1-\mathbb{E}[\Theta]) \\
&= 0.00009999 \\
\sigma(X) &= 0.0099995 \\
\mathbb{E}X_iX_j &= \mathbb{E}[X_iX_j|\Theta = \frac{1}{1\,000}]0.01 + \mathbb{E}[X_iX_j|\Theta = \frac{1}{11\,000}]0.99 \quad , \quad i \neq j \\
&= \left(\frac{1}{1\,000}\right)^2 0.01 + \left(\frac{1}{11\,000}\right)^2 0.99 \\
&= 1.8181\,10^{-8} \\
\mathbb{C}ov(X_i, X_j) &= \mathbb{E}[X_iX_j] - \mathbb{E}X_i\mathbb{E}X_j \quad , \quad i \neq j \\
&= 1.8181\,10^{-8} - \left(\frac{1}{10\,000}\right)^2 \\
&= 8.1818\,10^{-9} \\
\text{corr}(X_i, X_j) &= \frac{\mathbb{C}ov(X_i, X_j)}{\sigma(X_i)\sigma(X_j)} \quad , \quad i \neq j \\
&= 8.18264\,10^{-5}.
\end{aligned}$$

We observe that even though the dependency seems to be very low (correlation coefficient is close to 0), the effect on the required solvency level is very important. In fact the correlation coefficient is not a good measure for the level of dependency. It is indeed possible that highly dependent risks have a zero correlation coefficient. As an example, if $X \sim \text{Nor}(0,1)$, then $\mathbb{C}ov(X, X^2) = 0$ whereas X and X^2 are obviously highly dependent. See Embrechts et al. (2002) for more details.

In our case dependency is induced by extreme events. This is a typical situation financial conglomerates have to face within our globalized socio-economic environment. It is readily acknowledged that most of the risks are almost independent. However in the tail of the distributions we may observe stochastic dependencies, which are of particular interest when it comes to the calculation of the required solvency level.

A good example is the Kobe earthquake in 1995. After this earthquake the Asian stock markets crashed, showing the dependency between property insurance, cat nat events and stocks behaviour. On top of this, the positions on derivative assets taken by a trader of the Barings Bank in Singapore, sank the asset position of the bank due to the stock crash.

This puts in light the stochastic dependency between the previous risks and operational risk. The bad outcome of operational risk (in this case the lack of control within Barings bank) happened at the same time as the bad outcome of market and property insurance risks. That kind of extreme dependencies in the tails is nowadays recognized by copulae.

A copula is a function that links the distribution functions of different random variables within a stochastic dependency context. Copulas have been introduced a long time ago by Sklar (1959) and are gaining interest in actuarial science to model stochastic dependencies. The interested reader may consult e.g. Nelsen (1999) for a reference on copulas and Frees and Valdez (1998) for an introduction in an actuarial context. See also Embrechts et al. (2003). Summarizing we can say that modelling marginal distributions together with copulas provides a model for the aggregate portfolio accounting for dependencies between lines of business.

Dependencies may also be readily acknowledged in the actuarial model. Assume indeed extreme shocks that may be induced by our socio-economic environment. Assume that the chance of fire increases when the economy is down (due to fraudulent behaviour adopted by policyholders needing money from their insurer to recover from delicate financial situations: a real-life application of moral hazard behaviour). Assume that the economy is down with probability 1% whereas it is up with probability 99%. Assume that when the economy is down the aggregate claim is normally distributed with parameters $\mu = 1000$ and $\sigma = 500$ whereas when the economy is up, the aggregate claim is normally distributed with parameters $\mu = 500$ and $\sigma = 250$.

Our model is

$$S = \Theta S_1 + (1 - \Theta) S_2$$

where $S_1 \sim \text{Nor}(1000, 500)$, $S_2 \sim \text{Nor}(500, 250)$ and $\Theta \sim \text{Bin}(1, 1/100)$.

If we do not recognize these two states of nature, we may be tempted to work with an aggregate claims distributed as a normal distribution with parameters

$$\begin{aligned}
\mu &= \mathbb{E}[\mathbb{E}[S|\Theta]] \\
&= \mathbb{E}\Theta\mathbb{E}S_1 + \mathbb{E}(1-\Theta)\mathbb{E}S_2 \\
&= 505 \\
\sigma^2 &= \mathbb{E}[\mathbb{V}ar[S|\Theta]] + \mathbb{V}ar[\mathbb{E}[S|\Theta]] \\
&= \mathbb{E}\Theta^2\mathbb{V}ar S_1 + \mathbb{E}(1-\Theta)^2\mathbb{V}ar S_2 + (\mathbb{E}S_1 - \mathbb{E}S_2)^2\mathbb{V}ar\Theta \\
&= 258.55.
\end{aligned}$$

In table 5.1, we compare the required solvency level with the correct model that recognizes the stochastic dependency and the wrong model that ignores the stochastic dependency.

ϵ	75%	90%	95%	99%	99.9%	99.99%	99.999%
				VaR_ϵ			
Correct model	672.56	828.23	924.25	1 127.10	1 641.47	2 163.17	2 545.12
Wrong model	679.39	836.34	930.28	1 106.48	1 303.99	1 466.68	1 607.84
				TVaR_ϵ			
Correct model	832.54	967.55	1 063.36	1 317.76	1 877.61	2 332.61	2 683.44
Wrong model	833.64	958.75	1 038.32	1 194.11	1 375.54	1 526.75	1 660.19

Table 5.1: Comparison of VaR_ϵ and TVaR_ϵ with and without stochastic dependency.

Once again we observe that the introduction of dependencies in our model dramatically influences the required solvency level as soon as the security level is high. Note that for low security levels, the wrong model may be conservative.

Assume that the outcomes are perturbated by a random effect $1+V$ that is dependent on the state of the nature. Assume the following bivariate probability function

Θ/V	0.5	-0.5	0.2	-0.2
0	0	0	0.495	0.495
1	0.005	0.005	0	0

Table 5.2: Bivariate probability function of (Θ, V)

Then the model becomes

$$\begin{aligned} S &= (\Theta S_1 + (1-\Theta)S_2)(1+V) \\ &= \Theta(1+V)S_1 + (1-\Theta)(1+V)S_2. \end{aligned}$$

Due to the specific dependence structure between Θ and V, we may write the latter expression as

$$S|X,Y \sim \mathrm{Nor}(X,Y)$$

where

$X=x$	$Y=y$	$\mathbb{P}[X=x,Y=y]$
1 500	750	0.005
500	250	0.005
600	300	0.495
400	200	0.495

Table 5.3: Bivariate probability function of (X,Y)

We can now compute VaR and TVar for this model :

ϵ	95%	99%	99.9%	99.99%	99.999%
VaR_ϵ	998.84	1 260.33	2 131.26	3 040.31	3 658.63
TVaR_ϵ	1 188.21	1 549.83	2 549.86	3 315.68	3 877.50

Table 5.4: Comparison of VaR_ϵ and TVaR_ϵ for the model with stochastic dependency and with model risk.

These results should be compared to the figures of table 5.1. We observe that, when the security level is high, ignoring the model risk may have a dramatic effect.

6 The Threat of Mixtures Models

We have observed in the previous sections that parameter risk may be modelled by using mixture models. Now we prove that it is not possible to completely eliminate the risk by diversification when working with mixture models.

Our generic model is

$$S|\Theta \sim \mathrm{Bin}(n, \Theta).$$

If Θ is not random, then it is well known that

$$\frac{S}{\mathbb{E}S} \to_{law} 1 \text{ when } n \to \infty.$$

In other words volatility risk is completely diversified in an infinitely large portfolio.

Let us prove that

$$\frac{S}{\mathbb{E}S} \to_{law} \frac{\Theta}{\mathbb{E}\Theta} \text{ when } n \to \infty.$$

We have

$$
\begin{aligned}
\mathbb{E}\left[\left(\frac{S}{\mathbb{E}S}-\frac{\Theta}{\mathbb{E}\Theta}\right)^2\right] &= \mathbb{E}\left[\mathbb{E}\left[\left(\frac{S}{\mathbb{E}S}-\frac{\Theta}{\mathbb{E}\Theta}\right)^2|\Theta\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\left(\frac{S}{\mathbb{E}S}-\mathbb{E}\left[\frac{S}{\mathbb{E}S}|\Theta\right]\right)^2|\Theta\right]\right] \\
&= \mathbb{E}\left[\mathbb{V}ar\left[\frac{S}{\mathbb{E}S}|\Theta\right]\right] \\
&= \frac{\mathbb{E}\mathbb{V}ar[S|\Theta]}{(\mathbb{E}S)^2} \\
&= \frac{\int_0^1 n\theta(1-\theta)dF_\Theta(\theta)}{\left(\int_0^1 n\theta dF_\Theta(\theta)\right)^2} \\
&= \frac{1}{n}\frac{\int_0^1 \theta(1-\theta)dF_\Theta(\theta)}{\left(\int_0^1 \theta dF_\Theta(\theta)\right)^2} \\
&\to 0 \text{ when } n\to\infty.
\end{aligned}
$$

Using Markov inequality we obtain

$$
\begin{aligned}
\mathbb{P}\left[|\frac{S}{\mathbb{E}S}-\frac{\Theta}{\mathbb{E}\Theta}|>\epsilon\right] &\leq \frac{\mathbb{E}|\frac{S}{\mathbb{E}S}-\frac{\Theta}{\mathbb{E}\Theta}|}{\epsilon} \\
&\leq \frac{1}{\epsilon}\sqrt{\mathbb{E}\left[\left(\frac{S}{\mathbb{E}S}-\frac{\Theta}{\mathbb{E}\Theta}\right)^2\right]} \\
&\to 0 \text{ when } n\to\infty.
\end{aligned}
$$

From the previous inequality we conclude that

$$\frac{S}{\mathbb{E}S}\to_{proba}\frac{\Theta}{\mathbb{E}\Theta} \text{ when } n\to\infty.$$

Because convergence in probability implies convergence in law, we conclude that

$$\frac{S}{\mathbb{E}S}\to_{law}\frac{\Theta}{\mathbb{E}\Theta} \text{ when } n\to\infty.$$

Therefore even for very large portfolios the loss ratio remains stochastic and the volatility depends on the heterogeneity induced by Θ.

The result remains trivially true for

$$S|\Theta \sim \text{Nor}(n\Theta, \sqrt{n\Theta(1-\Theta)}).$$

This explains the sixth column of table 3.2 where

$$\mathrm{VaR}_{99.999\%}\left(\frac{\Theta}{\mathbb{E}\Theta}\right) = \frac{\frac{1}{1000}}{\frac{1}{10000}} = 10.$$

The same kind of result has been introduced in actuarial science by Beard et al. (1984) for the mixed compound Poisson model.

7 Multivariate Normal Model

We now move to an insurance company writing three kinds of risks. All the developments would remain true for n kinds of risks. We assume that these risks are normally distributed for the sake of simplicity. We recognize dependencies between these risks. We assume to have an amount of capital at disposal. Our aim will be to maximize the value creation for the shareholders. We only look at technical premiums, i.e. pure premiums plus capital charges. We disregard the management of administrative and acquisition expenses.

Assume the following types of normally distributed risks :

- n_1 risks of type 1 : $X_1, \ldots, X_{n_1}$
- n_2 risks of type 2 : $X_{n_1+1}, \ldots, X_{n_1+n_2}$
- n_3 risks of type 3 : $X_{n_1+n_2+1}, \ldots, X_{n_1+n_2+n_3}$.

Assume the following "identically distributed" assumptions

$$\begin{aligned}
\mu_{X_j} &= \mu_1 \,,\; j = 1, \ldots, n_1 \\
\mu_{X_j} &= \mu_2 \,,\; j = n_1+1, \ldots, n_1+n_2 \\
\mu_{X_j} &= \mu_3 \,,\; j = n_1+n_2+1, \ldots, n_1+n_2+n_3 \\
\sigma_{X_j} &= \sigma_1 \,,\; j = 1, \ldots, n_1 \\
\sigma_{X_j} &= \sigma_2 \,,\; j = n_1+1, \ldots, n_1+n_2 \\
\sigma_{X_j} &= \sigma_3 \,,\; j = n_1+n_2+1, \ldots, n_1+n_2+n_3 \\
\rho_{X_i,X_j} &= \rho_{11} \,,\; i = 1, \ldots, n_1, j = i+1, \ldots, n_1 \\
\rho_{X_i,X_j} &= \rho_{22} \,,\; i = n_1+1, \ldots, n_1+n_2, j = i+1, \ldots, n_1+n_2 \\
\rho_{X_i,X_j} &= \rho_{33} \,,\; i = n_1+n_2+1, \ldots, n_1+n_2+n_3, j = i+1, \ldots, n_1+n_2+n_3 \\
\rho_{X_i,X_j} &= \rho_{12} \,,\; i = 1, \ldots, n_1, j = n_1+1, \ldots, n_1+n_2 \\
\rho_{X_i,X_j} &= \rho_{13} \,,\; i = 1, \ldots, n_1, j = n_1+n_2+1, \ldots, n_1+n_2+n_3 \\
\rho_{X_i,X_j} &= \rho_{23} \,,\; i = n_1+1, \ldots, n_1+n_2, j = n_1+n_2+1, \ldots, n_1+n_2+n_3 \\
P_j &= (1+\alpha_j)n_j\mu_j.
\end{aligned}$$

where μ_{X_j} denotes the expectation of X_j, σ_{X_j} denotes the standard deviation of X_j and ρ_{X_i,X_j} denotes the correlation between X_i and X_j.

P_j denotes the total premium charged for policies of the line of business (LOB) j. The latter equation means that the average loading charged on policies of LOB j is α_j.

The variance-covariance matrix is the $(n_1+n_2+n_3)\times(n_1+n_2+n_3)$ matrix $\mathbf{\Sigma}$ with elements

$$\begin{aligned}\mathbf{\Sigma}(j,j) &= \sigma^2_{X_j}\,,\ 1\le j\le n_1+n_2+n_3,\\ \mathbf{\Sigma}(i,j) &= \rho_{X_i,X_j}\sigma_{X_i}\sigma_{X_j}\,,\ 1\le i\ne j\le n_1+n_2+n_3.\end{aligned}$$

In order to show the particular structure of the variance-covariance matrix in our example, we draw it for $n_1=n_2=n_3=3$:

$$\mathbf{\Sigma}=\begin{pmatrix}
\sigma_1^2 & \rho_{11}\sigma_1^2 & \rho_{11}\sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \rho_{12}\sigma_1\sigma_2 & \rho_{12}\sigma_1\sigma_2 & \rho_{13}\sigma_1\sigma_3 & \rho_{13}\sigma_1\sigma_3 & \rho_{13}\sigma_1\sigma_3\\
\rho_{11}\sigma_1^2 & \sigma_1^2 & \rho_{11}\sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \rho_{12}\sigma_1\sigma_2 & \rho_{12}\sigma_1\sigma_2 & \rho_{13}\sigma_1\sigma_3 & \rho_{13}\sigma_1\sigma_3 & \rho_{13}\sigma_1\sigma_3\\
\rho_{11}\sigma_1^2 & \rho_{11}\sigma_1^2 & \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \rho_{12}\sigma_1\sigma_2 & \rho_{12}\sigma_1\sigma_2 & \rho_{13}\sigma_1\sigma_3 & \rho_{13}\sigma_1\sigma_3 & \rho_{13}\sigma_1\sigma_3\\
\rho_{12}\sigma_1\sigma_2 & \rho_{12}\sigma_1\sigma_2 & \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 & \rho_{22}\sigma_2^2 & \rho_{22}\sigma_2^2 & \rho_{23}\sigma_2\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \rho_{23}\sigma_2\sigma_3\\
\rho_{12}\sigma_1\sigma_2 & \rho_{12}\sigma_1\sigma_2 & \rho_{12}\sigma_1\sigma_2 & \rho_{22}\sigma_2^2 & \sigma_2^2 & \rho_{22}\sigma_2^2 & \rho_{23}\sigma_2\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \rho_{23}\sigma_2\sigma_3\\
\rho_{12}\sigma_1\sigma_2 & \rho_{12}\sigma_1\sigma_2 & \rho_{12}\sigma_1\sigma_2 & \rho_{22}\sigma_2^2 & \rho_{22}\sigma_2^2 & \sigma_2^2 & \rho_{23}\sigma_2\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \rho_{23}\sigma_2\sigma_3\\
\rho_{13}\sigma_1\sigma_3 & \rho_{13}\sigma_1\sigma_3 & \rho_{13}\sigma_1\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \sigma_3^2 & \rho_{33}\sigma_3^2 & \rho_{33}\sigma_3^2\\
\rho_{13}\sigma_1\sigma_3 & \rho_{13}\sigma_1\sigma_3 & \rho_{13}\sigma_1\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \rho_{33}\sigma_3^2 & \sigma_3^2 & \rho_{33}\sigma_3^2\\
\rho_{13}\sigma_1\sigma_3 & \rho_{13}\sigma_1\sigma_3 & \rho_{13}\sigma_1\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \rho_{33}\sigma_3^2 & \rho_{33}\sigma_3^2 & \sigma_3^2
\end{pmatrix}.$$

Because $\mathbf{\Sigma}$ is a variance-covariance matrix, it is important to make sure that $\mathbf{\Sigma}$ is positive semidefinite, i.e. that

$$\boldsymbol{a}^t\cdot\mathbf{\Sigma}\cdot\boldsymbol{a}\ge 0$$

for all nonzero $(n_1+n_2+n_3)\times 1$ vectors $\boldsymbol{a}$.

A necessary and sufficient condition for positive semidefiniteness is that all the eigenvalues of $\mathbf{\Sigma}$ are nonnegative. This is the condition we will verify. The reader is referred to e.g. Horn and Johnson (1985) and Golub and Van Loan (1996) for more details on matrix analysis. All matrix theory results used in this paper may be found in appendix A.

The particular structure of our variance-covariance matrix may be summarized with the following notation :

$$\mathbf{\Sigma}=\begin{pmatrix}\boldsymbol{D}_{11} & \boldsymbol{D}_{12} & \boldsymbol{D}_{13}\\ \boldsymbol{D}_{21} & \boldsymbol{D}_{22} & \boldsymbol{D}_{23}\\ \boldsymbol{D}_{31} & \boldsymbol{D}_{32} & \boldsymbol{D}_{33}\end{pmatrix}$$

where $\boldsymbol{D}_{pp}$ is a $n_p\times n_p$ matrix such that

$$\begin{aligned}\boldsymbol{D}_{pp}(i,i) &= \sigma_p^2\,,\ 1\le i\le n_p,\\ \boldsymbol{D}_{pp}(i,j) &= \rho_{pp}\sigma_p^2\,,\ 1\le i\ne j\le n_p\end{aligned}$$

and $\boldsymbol{D}_{pq}$ is a $n_p\times n_q$ matrix such that

$$\boldsymbol{D}_{pq}(i,j) = \rho_{pq}\sigma_p\sigma_q\,,\ 1\le i\le n_p\,,\ 1\le j\le n_q.$$

Let us denote by $\boldsymbol{S}$ the diagonal matrix with diagonal elements $(\underbrace{\sigma_1, \ldots, \sigma_1}_{n_1 \text{ times}}, \underbrace{\sigma_2, \ldots, \sigma_2}_{n_2 \text{ times}}, \underbrace{\sigma_3, \ldots, \sigma_3}_{n_3 \text{ times}})$.

Let $\boldsymbol{R}$ be

$$\boldsymbol{R} = \begin{pmatrix} \boldsymbol{R}_{11} & \boldsymbol{R}_{12} & \boldsymbol{R}_{13} \\ \boldsymbol{R}_{21} & \boldsymbol{R}_{22} & \boldsymbol{R}_{23} \\ \boldsymbol{R}_{31} & \boldsymbol{R}_{32} & \boldsymbol{R}_{33} \end{pmatrix}$$

where $\boldsymbol{R}_{pp}$ is a $n_p \times n_p$ matrix such that

$$\begin{aligned} \boldsymbol{R}_{pp}(i,i) &= 1\,,\ 1 \le i \le n_p, \\ \boldsymbol{R}_{pp}(i,j) &= \rho_{pp}\,,\ 1 \le i \ne j \le n_p \end{aligned}$$

and $\boldsymbol{R}_{pq}$ is a $n_p \times n_q$ matrix such that

$$\boldsymbol{R}_{pq}(i,j) = \rho_{pq}\,,\ 1 \le i \le n_p\,,\ 1 \le j \le n_q.$$

We have

$$\boldsymbol{\Sigma} = \boldsymbol{S} \cdot \boldsymbol{R} \cdot \boldsymbol{S}^t.$$

Because $\boldsymbol{S}$ is nonsingular (we indeed exclude degenerate cases where $\sigma = 0$), $\boldsymbol{\Sigma}$ is congruent to $\boldsymbol{R}$. This implies that if the eigenvalues of $\boldsymbol{R}$ are nonnegative, then the eigenvalues of $\boldsymbol{\Sigma}$ are nonnegative as well. Indeed symmetric and congruent matrices have the same inertia according to Sylvester's law of inertia. The problem is thus reduced to the analysis of the sign of the eigenvalues of $\boldsymbol{R}$. The values of σ_i do not play a role in our problem.

Let $\boldsymbol{U}$ be

$$\boldsymbol{U} = \begin{pmatrix} \boldsymbol{U}_1 & 0 & 0 \\ 0 & \boldsymbol{U}_2 & 0 \\ 0 & 0 & \boldsymbol{U}_3 \end{pmatrix}$$

where $\boldsymbol{U}_i$ is a $n_i \times n_i$ matrix such that

$$\boldsymbol{U}_i \cdot \mathbf{1}_{n_i} = (\sqrt{n_i}, \underbrace{0, \ldots, 0}_{n_i - 1 \text{ times}})^t$$

with $\mathbf{1}_{n_i}$ a $n_i \times 1$ vector with all entries equal to 1. An appropriate candidate for $\boldsymbol{U}_i$ is the Householder transform.

We have that $\boldsymbol{T}$ is congruent to $\boldsymbol{R}$:

$$\boldsymbol{U} \cdot \boldsymbol{R} \cdot \boldsymbol{U}^t = \boldsymbol{T} = \begin{pmatrix} \boldsymbol{T}_{11} & \boldsymbol{T}_{12} & \boldsymbol{T}_{13} \\ \boldsymbol{T}_{21} & \boldsymbol{T}_{22} & \boldsymbol{T}_{23} \\ \boldsymbol{T}_{31} & \boldsymbol{T}_{32} & \boldsymbol{T}_{33} \end{pmatrix}$$

where $\boldsymbol{T}_{pp}$ is a $n_p \times n_p$ matrix such that

$$\begin{aligned} \boldsymbol{T}_{pp}(i,i) &= 1 - \rho_{pp} + \rho_{pp} n_p\,,\ 1 \le i \le n_p \\ \boldsymbol{T}_{pp}(i,j) &= 0\,,\ 1 \le i \ne j \le n_p \end{aligned}$$

and $\boldsymbol{T}_{pq}$ is a $n_p \times n_q$ matrix such that

$$\begin{aligned} \boldsymbol{T}_{pq}(1,1) &= \rho_{pq}\sqrt{n_p n_q}, \\ \boldsymbol{T}_{pq}(i,j) &= 0 \text{ else }. \end{aligned}$$

Using an adequate permutation matrix $\boldsymbol{P}$, we have that $\boldsymbol{V}$ is congruent to $\boldsymbol{T}$:

$$\boldsymbol{P} \cdot \boldsymbol{T} \cdot \boldsymbol{P}^t = \boldsymbol{V} = \begin{pmatrix} \boldsymbol{X} & \mathbf{0}_{3\times n_1} & \mathbf{0}_{3\times n_2} & \mathbf{0}_{3\times n_3} \\ \mathbf{0}_{n_1\times 3} & \boldsymbol{W}_1 & \mathbf{0}_{n_1\times n_2} & \mathbf{0}_{n_1\times n_3} \\ \mathbf{0}_{n_2\times 3} & \mathbf{0}_{n_2\times n_1} & \boldsymbol{W}_2 & \mathbf{0}_{n_2\times n_3} \\ \mathbf{0}_{n_3\times 3} & \mathbf{0}_{n_3\times n_1} & \mathbf{0}_{n_3\times n_2} & \boldsymbol{W}_3 \end{pmatrix}$$

where $\boldsymbol{X}$ is a 3×3 matrix :

$$\boldsymbol{X} = \begin{pmatrix} (n_1-1)\rho_{11}+1 & \sqrt{n_1 n_2}\rho_{12} & \sqrt{n_1 n_3}\rho_{13} \\ \sqrt{n_1 n_2}\rho_{12} & (n_2-1)\rho_{22}+1 & \sqrt{n_2 n_3}\rho_{23} \\ \sqrt{n_1 n_3}\rho_{13} & \sqrt{n_2 n_3}\rho_{23} & (n_3-1)\rho_{33}+1 \end{pmatrix},$$

$\boldsymbol{W}_i$ is a $n_i \times n_i$ diagonal matrix with diagonal elements all equal to $1-\rho_{ii}$ and $\mathbf{0}_{u,v}$ is a $u \times v$ matrix with all entries equal to 0.

We then have nonnegative eigenvalues if :

- $\rho_{ii} \leq 1$
- the eigenvalues of the matrix $\boldsymbol{X}$ are nonnegative.

We therefore only have to make sure that the eigenvalues of $\boldsymbol{X}$ are nonnegative.

Let $\boldsymbol{F}$ be a 3×3 diagonal matrix with diagonal elements $(1/\sqrt{n_1}, 1/\sqrt{n_2}, 1/\sqrt{n_3})$. We have that $\boldsymbol{X}$ and $\boldsymbol{Z}$ are congruent :

$$\boldsymbol{F} \cdot \boldsymbol{X} \cdot \boldsymbol{F}^t = \boldsymbol{Z} = \begin{pmatrix} \rho_{11} + \frac{1-\rho_{11}}{n_1} & \rho_{12} & \rho_{13} \\ \rho_{12} & \rho_{22} + \frac{1-\rho_{22}}{n_2} & \rho_{23} \\ \rho_{13} & \rho_{23} & \rho_{33} + \frac{1-\rho_{33}}{n_3} \end{pmatrix}.$$

So we have to compute the eigenvalues of $\boldsymbol{Z}$ and make sure that they are nonnegative. This is equivalent to verifying that all principal determinants of $\boldsymbol{Z}$ are nonnegative.

We would like to obtain sufficient conditions for positive semidefiniteness for all values of n_1, n_2 and n_3.

After some algebra (see appendix B for details), we obtain the following sufficient conditions:

$$
\begin{aligned}
\rho_{11}\rho_{22}\rho_{33} + 2\rho_{12}\rho_{13}\rho_{23} &\geq \rho_{11}\rho_{23}^2 + \rho_{22}\rho_{13}^2 + \rho_{33}\rho_{12}^2 \\
\rho_{11}\rho_{22} &\geq \rho_{12}^2 \\
\rho_{11}\rho_{33} &\geq \rho_{13}^2 \\
\rho_{22}\rho_{33} &\geq \rho_{23}^2 \\
\rho_{11} &\geq 0 \\
\rho_{22} &\geq 0 \\
\rho_{33} &\geq 0
\end{aligned}
$$

Not imposing these conditions may lead to impossible situations. Assume indeed the following parameters :

$$
\begin{aligned}
\mu_1 = \mu_2 = \mu_3 &= 1 \\
\sigma_1 = \sigma_2 = \sigma_3 &= 1 \\
\rho_{11} = \rho_{22} = \rho_{33} &= 0.1 \\
\rho_{12} = \rho_{13} &= 0.1 \\
\rho_{23} &= 0.2
\end{aligned}
$$

Then the sufficient conditions are not fulfilled for all values of n_1, n_2 and n_3 because

$$\rho_{22}\rho_{33} = 0.01 < 0.04 = \rho_{23}^2.$$

For $n_1 = n_2 = n_3 = 2$, the matrix is positive semidefinite whereas for $n_1 = n_2 = n_3 = 5$ it is not. It is however clear that the model must be valid for any size of the portfolio.

This clearly shows that working with multivariate models accounting for dependencies is a very complicated task. It also shows that accounting for dependencies between lines of business may not be coherent as soon as these lines of business may increase in size.

The required solvency level is given by the Tail Value at Risk at level ϵ of the random variable

$$S = \underbrace{X_1 + \cdots + X_{n_1}}_{S_1} + \underbrace{X_{n_1+1} + \cdots + X_{n_1+n_2}}_{S_2} + \underbrace{X_{n_1+n_2+1} + \cdots + X_{n_1+n_2+n_3}}_{S_3},$$

where S denotes the aggregate claims of the insurer. S is normally distributed with parameters

$$
\begin{aligned}
\mu_S &= \sum_{i=1}^{3} n_i \mu_i \\
\sigma_S^2 &= \sum_{i=1}^{3} n_i \sigma_i^2 + 2\sum_{i=2}^{3} n_i n_1 \rho_{1i}\sigma_1\sigma_i + 2n_2 n_3 \rho_{23}\sigma_2\sigma_3 + \sum_{i=1}^{3} n_i(n_i - 1)\rho_{ii}\sigma_i^2.
\end{aligned}
$$

S_i is normally distributed with parameters

$$\begin{aligned}\mu_{S_i} &= n_i\mu_i \\ \sigma^2_{S_i} &= n_i\sigma_i^2 + n_i(n_i-1)\rho_{ii}\sigma_i^2.\end{aligned}$$

The covariance between S_i and S_j is given by

$$\mathbb{C}ov(S_i, S_j) = n_i n_j \rho_{ij}\sigma_i\sigma_j.$$

The correlation between S_i and S_j is given by

$$\rho_{S_i,S_j} = \frac{n_i n_j \rho_{ij}}{\sqrt{n_i(1+(n_i-1)\rho_{ii})}\sqrt{n_j(1+(n_j-1)\rho_{jj})}}.$$

The total premium charged by the insurance company is $P = P_1+P_2+P_3$ including a margin equal to

$$Margin = P - \mu = \sum_{i=1}^{3} \alpha_i n_i \mu_i.$$

The risk adjusted capital is then

$$RAC = RSL - P.$$

Recall that we have chosen to work with

$$RSL = TVaR_\epsilon(S).$$

The EVA (economic value added) is given by

$$EVA = Margin - kRAC$$

where k is the cost of capital demanded by the shareholders. We may also compute the RORAC ((expected) return on risk adjusted capital):

$$RORAC = \frac{Margin}{RAC}.$$

Our aim will be to maximize the EVA under a capital constraint. Moreover we will also look at the EVA per line of business. This is often done in order to assess the profitability of a given line of business and the financial compensation of its manager is usually linked to the calculated profitability, see also Valdez and Chernih (2003) for a discussion. We will observe that this type of reasoning may be dangerous.
Now we need to allocate the full capital between the lines of business. This will be done using the Tail Value at Risk allocation method :

$$RAC_i = \mathbb{E}[S_i|S > VaR_\epsilon(S)] - P_i$$

where S_i denotes the aggregate claims for LOB i and P_i denotes the premium for LOB i.

This allocation method is additive :

$$RAC = RAC_1 + RAC_2 + RAC_3.$$

Moreover this allocation respects the no undercut axiom (see Denault (2001) for more details). Indeed using this allocation method together with a subadditive risk measure ensures that coalitions will not be favoured. For example lines of business 1 and 2 will never have an incentive to leave the conglomerate due to the fact that they would consume less capital on a stand alone basis than within the full conglomerate. This no undercut axiom is not verified for a range of classical allocation methods such as the relative allocation :

$$RAC_i = RAC \frac{K_i}{K_1 + K_2 + K_3}$$

where K_i is the stand alone RAC of line of business i :

$$K_i = \mathbb{E}[S_i | S > VaR_\epsilon(S)] - P_i.$$

When the risks are normally distributed, we deduce the following formula from Panjer (2001):

$$RAC_i = \sigma_{S_i} \sigma_S \rho_{S_i,S} \frac{\phi(\Phi^{-1}(\epsilon))}{1-\epsilon} - Margin_i$$

where $\rho_{S_i,S}$ denotes the correlation between S_i and S.

It then becomes easy to analyse how the value is created within the business lines :

$$EVA_i = \text{Margin}_i - kRAC_i.$$

One may also be tempted to compute the RORAC per line of business :

$$RORAC_i = \frac{\text{Margin}_i}{RAC_i}.$$

We will see that this measure may be extremely misleading.

Let us move to numerical applications. Assume the following parameters :

$$\begin{aligned}
\mu_1 = \mu_2 = \mu_3 &= 1 \\
\sigma_1 = \sigma_2 = \sigma_3 &= 1 \\
\rho_{11} &= 0.1 \\
\rho_{22} &= 0.1 \\
\rho_{33} &= 0.1 \\
\rho_{12} &= -0.01 \\
\rho_{13} &= -0.01 \\
\rho_{23} &= 0.01 \\
\alpha_1 = \alpha_2 = \alpha_3 &= 0.10 \\
\epsilon &= 99\% \\
\text{Cost of capital} &= 15\% \\
\text{Available capital} &= 100.
\end{aligned}$$

One easily checks that the sufficient conditions for positive semidefiniteness are verified.

We now look for the optimal number of risks to write in order to maximize the value creation for the shareholders. We find :

LOB	n	RAC	Margin	EVA	RORAC
1	94	37.09	9.4	3.84	25.34%
2	80	31.75	8	3.24	25.20%
3	79	30.95	7.9	3.26	25.53%
Total	253	99.79	25.3	10.33	25.35%

Table 7.1: Initial portfolio.

LOB 1 is favoured because it is negatively correlated with LOB 2 and LOB 3.

Now we change the loading of LOB 1 from 10% to −1%. We also change $\rho_{12} = \rho_{13}$ from −0.01 to −0.02 (sufficient conditions are still verified). We obtain

LOB	n	RAC	Margin	EVA	RORAC
1	24	−0.37	−0.24	−0.19	65.72%
2	93	50.16	9.3	1.78	18.54%
3	93	50.16	9.3	1.78	18.54%
Total	210	99.96	18.36	3.37	18.37%

Table 7.2: LOB 1 with loading −1% and more negatively correlated with LOB 2 and LOB 3.

We observe that it remains interesting to write risks from LOB 1 even though their margin is negative. Indeed these risks allow for an excellent diversification within the conglomerate and they in fact lend capital to the other LOB's. We also conclude that the negative EVA for LOB 1 is not informative. The RORAC for this LOB is extremely difficult to analyze.

Let us compute the EVA for the case where we would abandon LOB 1 :

LOB	n	RAC	Margin	EVA	RORAC
1	0	0	0	0	
2	90	49.76	9	1.54	18.09%
3	90	49.76	9	1.54	18.09%
Total	180	99.51	18	3.07	18.09%

Table 7.3: LOB 1 disregarded.

Clearly the value creation is now lower ($3.07 < 3.37$) than with LOB 1 within the conglomerate.

Now let us come back to the initial situation but assume the available capital is 200 instead of 100 :

LOB	n	RAC	Margin	EVA	RORAC
1	194	73.54	19.4	8.37	26.38%
2	165	63.06	16.5	7.04	26.17%
3	165	63.06	16.5	7.04	26.17%
Total	524	199.66	52.4	22.45	26.24%

Table 7.4: Increased available capital.

Table 7.4 shows that the value creation for the shareholder is now more than twice the value creation with available capital equal to 100. This is obviously due to the diversification benefit. Because the risks are not perfectly dependent (comonotonic) there is a relative gain in terms of consumed capital.

Now let us come back to the initial situation and let us assume that the risks of LOB 1 have the following adapted characteristics : $\mu_1 = 2$ and $\sigma_1 = 2$:

LOB	n	RAC	Margin	EVA	RORAC
1	45	38.35	9	3.25	23.47%
2	79	31.19	7.9	3.22	25.33%
3	78	30.39	7.8	3.24	25.67%
Total	202	99,93	24.7	9.71	24.72%

Table 7.5: LOB 1 with larger risks.

Table 7.5 illustrates that the value creation is now lower than with the initial portfolio because the large risks consume a lot of capital. There is less diversification.

Finally, let us come back to the initial situation and assume that the conglomerate buys unlimited stop-loss reinsurance in excess of d. It is well known that reinsurance is an alternative to holding economic capital. It is therefore interesting to analyze the decision of buying reinsurance.

The pure premium for such reinsurance is given by

$$PP^{Re} = \sigma_S \phi \left(\frac{d - \mu_S}{\sigma_S} \right) - (d - \mu_S) \left(1 - \Phi \left(\frac{d - \mu_S}{\sigma_S} \right) \right).$$

Assume a reinsurance loading β. The commercial reinsurance premium is then $P^{Re} = (1 + \beta)PP^{Re}$. Let us compute the stop-loss premium for a threshold $d = VaR_{99\%}(S)$. We denote the retention of the insurance company by S^{Ret}. We now compare the EVA and RORAC for different loadings charged by the reinsurer. We do not analyze the EVA per line of business because we would have to resort to simulations to do so and it is part of our conclusion that looking at EVA per LOB is not optimal. In order to keep things simple, we exclude the possible default of the reinsurer. Note that the risk of default of the reinsurer is positively dependent with the risk of hitting the reinsurance cover. This would immediately lead to complicated models. We find

$$\begin{aligned}
n_1 &= 94 \\
n_2 &= 80 \\
n_3 &= 79 \\
P &= 278.3 \\
Margin(S) &= 25.3 \\
d = VaR_{99\%}(S) &= 362.17 \\
PP^{Re} &= 0.1590 \\
P^{Re} &= (1+\beta)PP^{Re} \\
VaR_{99\%}(S^{Ret}) &= 362.17 \\
TVaR_{99\%}(S^{Ret}) &= 362.17 \\
RAC(S^{Ret}) &= 362.17 - (278.3 - P^{Re}) \\
Margin(S^{Ret}) &= 25.3 - \beta PP^{Re} \\
EVA(S^{Ret}) &= Margin(S^{Ret}) - kRAC(S^{Ret}) \\
RORAC(S^{Ret}) &= \frac{Margin(S^{Ret})}{RAC(S^{Ret})}.
\end{aligned}$$

β	No reinsurance	1	4	9	14	19
$RAC(S^{Ret})$	99.79	84.2	84.68	85.47	86.27	87.06
$EVA(S^{Ret})$	10.33	12.51	11.96	11.04	10.13	9.21
$RORAC(S^{Ret})$	25.35%	29.86%	29.13%	27.92%	26.74%	25.59%

Table 7.6: Performance with reinsurance.

We observe that even with a substantial loading charged by the reinsurer, it may be interesting to buy the reinsurance cover. Now we may do even better. Indeed, we have some capital left thanks to the reinsurance. For that reason, it is also meaningless to draw conclusions on basis of EVA. RORAC is a good tool in this case. What we now are going to do is to optimize the number of policies to write knowing that we buy the reinsurance cover. We obtain the table 7.7 :

β	No reinsurance	1	4	9	14	19
n_1	94	113	112	111	110	109
n_2	80	95	95	94	93	92
n_3	79	95	95	94	93	92
$d = VaR_{99\%}(S)$	362.18	432.64	431.23	427.00	422.77	418.55
$RAC(S^{Ret})$	99.79	99.72	99.97	99.97	99.95	99.91
$EVA(S^{Ret})$	10.33	15.15	14.45	13.22	12.02	10.83
$RORAC(S^{Ret})$	25.35%	30.20%	29.45%	28.23%	27.03%	25.85%

Table 7.7: Optimal underwriting with reinsurance.

We conclude that the underwriting decision is influenced by the reinsurance cover bought by the insurer.

8 Extensions to the Multivariate Normal Model

In practice insurance risks are not normally distributed. Most of the time they present fatter tails than the tails of normal distributions. A possible extension of the multivariate normal distribution is then to work with multivariate elliptic distributions. See e.g. Valdez and Chernih (2003) for recent discussions in an actuarial context.

Another option is to resort to copulas. A copula is a function that links the distribution functions of different random variables within a stochastic dependency context. Copulas have been introduced a long time ago by Sklar (1959) and are gaining interest in actuarial science to model stochastic dependencies. The interested reader may consult e.g. Nelsen (1999) for a reference on copulas and Frees and Valdez (1998) for an introduction in an actuarial context. See also Embrechts et al. (2003). Summarizing we can say that modelling marginal distributions together with copulas provides a model for the aggregate portfolio accounting for dependencies between lines of business.

Copulas are excellent means in order to make sensitivity analyses about the stochastic dependency and their implications on e.g. capital allocation matters. However, they remain computationnally difficult to use in multiple dimensions. Therefore one may not be able to provide a full modelling like the multivariate normal model presented in the previous section. The best one can do with copulas is to model the stochastic dependency between lines of business.

It is also worth noting that our socio-economic environment plays a major role in the determination of stochastic dependencies. When inflation is high, claim costs in workers compensation and motor third party liability are simultaneously high. On the other hand financial revenues increase, which is a partial hedge against the underwriting loss resulting from the high claim costs. When the economy is up, we have less fire, less default and less disability claims. We have the opposite when the economy is down. Therefore it is probably most interesting to develop economic environment generators allowing for a lot of stochastic dependencies within financial conglomerates.

9 Conclusion

It is clear that regulators have to acknowledge the diversification benefit much better than the EU rules suggest. Nevertheless it is also clear that all the risks have to be taken into account when building internal models. In particular, studying volatility risk given a model and parameters is rather dangerous unless the model has been deliberately chosen on the safe side. Accounting for model and parameter risks is of paramount importance and will lead to very complicated discussions between regulators and financial conglomerates. Another topic that will be difficult to handle is the notion of stochastic dependencies. Actuaries now have the tools in order to make sensitivity analyses on that topic. However it appears that we lack statistical material in order to have a good idea about the level of stochastic dependencies. Model and parameter risks are still there.

In times where the emphasis is put on corporate governance and on the restoring of investor's confidence (or better all stakeholder's confidence), it should be expected that financial conglomerates and regulators adopt very transparent ways of working, including communicating.

In some cases writing insurance business may turn out to be too expensive in terms of economic capital when the diversification effect is too low. This is in particular the case for lines of business where the model risk is huge and where we have potential very big accumulations. Cat nat perils and annuities (longevity risk) are two main examples. These perils may not be borne by insurers and reinsurers because of the lack of diversification opportunities. Securitization then arises as an alternative in order to cover the policyholders. In such a case primary insurers and reinsurers become risk transfer vehicles to the financial markets.

Model risk is a key issue for analysing natural catastrophe coverages. To model these, insurers essentially use different types of commercial software modelling storm, earthquake and flood risk. There is a strong model risk associated with the models used in this software. Think e.g. of earthquake where there are very few data which makes it extremely difficult to forecast what the effect of a major earthquake might be on buildings. Think also of the Lothaer and Martin storms in December 1999 in France. After these storms the software has been strongly updated because it did not account for such extreme events. Another example is the tsunami of December 2004 in the Indian Ocean. That kind of risk was not modelled before it did happen. Strong discussions will be needed with the regulators in order to agree upon acceptable models to be used in order to obtain the required solvency level.

Froot and Posner (2002) have analyzed the parameter uncertainty related to the pricing of cat bonds from the point of view of the buyer of the bond. They argue that the existing spread obtained on such cat bonds cannot be explained by parameter uncertainty on the probability of occurrence of the triggering event. This is not in contradiction with our findings. Indeed, Froot and Posner (2002) do not look at the tails of the claims distribution for their pricing purpose. Therefore they do not have to deal with highly nonlinear models and their parameter uncertainty does not play a major role on the pricing of their cat bonds. Indeed $\mathbb{E}(X_1 + \cdots + X_n) = \mathbb{E}X_1 + \cdots + \mathbb{E}X_n$ even when the $X_i' s$ are dependent.

As explained by Froot and Posner (2002), the spread may be explained by the fact that investors do not believe in the event probabilities deduced by the cat modelling firms. Investors would believe that there is a downward bias on the event probability. Froot and Posner (2002) do not believe that investors are more clever than the models. I however think that investors charge the issuer of the bond with a loading for agency costs. It is clear that the modelling of cat losses is extremely opaque to the investor. Therefore the investor likes to charge a fee on top of the normal actuarial price. This is somewhere related to the parameter and model uncertainty, which the investor may believe to be of importance.

Note that stochastic dependencies may also be introduced by the use of specific reinsurance covers. See e.g. Walhin (2002), Witdouck and Walhin (2004) or Walhin and Denuit (2005).

We have analyzed that lines of business presenting negative margins may indeed be profitable for the whole conglomerate due to possible negative dependencies (or in fact low dependencies) between some lines of business. This goes against the traditional belief that managers must be remunerated according to the performance of their business unit. In fact, it may happen that the EVA of a particular LOB is negative. Nevertheless the manager should be rewarded if he has been able to reach the goals defined a priori in case these goals include writing policies of that LOB. Managers should be rewarded in function of the a priori objective, in our case the number of policies to write. This is true under the hypothesis that they write business at average market conditions. Then they should be rewarded in function of the global value creation. Only when managers beat the market they should be rewarded on a local view. This happens when a manager is able to obtain higher margins than the market. Once again, looking at a LOB individually without accounting for the possible dependencies with the financial conglomerate is not optimal. The decision to invest or to desinvest should be taken after a global calculation.

In section 7 we decided to disregard the effect of administrative expenses. However, it is clear that the decision to invest in one line of business or another cannot be disconnected from the effective administrative expenses incurred when writing these lines of business. The actual value creation for the shareholders is equal to Margin - capital charges - administrative expenses. The latter may have a dramatic influence on the decision to invest and requires a very close management. Indeed it would not make sense to apply very sophisticated models for the calculation of the margin and allocated capital when administrative expenses are roughly estimated.

References

[1] Artzner, P., Delbaen, F., Eber, J.M. and Heath, D. (1999), Coherent Measures of Risk. Mathematical Finance, 9, 203–228.

[2] Beard, R.E., Pentikäinen, T. and Pesonen, M. (1984), Risk Theory. Chapman & Hall, London.

[3] Denault, M. (2001), Coherent Allocation of Risk Capital. Journal of Risk, 4, 7–21.

[4] Denuit, M., Dhaene, J., Goovaerts, M. and Kaas, R. (2005) Actuarial Theory for Dependent Risks, Wiley.

[5] Embrechts, P., Lindskog, F., McNeil, A. (2003) Modelling Dependence with Copulas and Applications to Risk Management. In: Handbook of Heavy Tailed Distributions in Finance, ed. S. Rachev, Elsevier, Chapter 8, pp. 329-384.

[6] Embrechts, P., McNeil, A., Straumann, D. (2002) Correlation and dependence in risk management: properties and pitfalls. In: Risk Management: Value at Risk and Beyond, ed. M.A.H. Dempster, Cambridge University Press, Cambridge, pp. 176-223.

[7] Frees, E.W. and E.A. Valdez. (1998) Understanding Relationships Using Copulas. North American Actuarial Journal 2, 1-25.

[8] Froot, K.A. and S. Posner. (2002) The Pricing of Event Risks with Parameter Uncertainty. The Geneva Papers on Risk and Insurance 27, no. 2 (December 2002): 153-165.

[9] Golub, G.H. and Van Loan, C.F. (1985) Matrix Computations. Johns Hopkins University Press, Baltimore.

[10] Horn, R.A. and Johnson, C.A. (1985) Matrix Analysis. Cambridge University Press, Cambridge.

[11] International Actuarial Assocation (2004) A Global Framework for Insurer Solvency Assessment. Report of the Insurer Solvency Assessment Working Party.

[12] Nelsen, R. (1999), An Introduction to Copulas. Springer, New York.

[13] Panjer, H. (2001), Measurement of Risk, Solvency Requirements and Allocation of Capital within Financial Conglomerates. Research Report 01-15, Institute of Insurance and Pension Research, University of Waterloo.

[14] Sklar, A. (1959), Fonctions de répartition à n dimensions et leurs marges. Publications de l'Institut de Statistique de l'Université de Paris 8, 229-231.

[15] Valdez, E.A. and Chernih, A. (2003), Wang's Capital Allocation Formula for Elliptically Contoured Distributions. Insurance:Mathematics & Economics, 33, 517–532.

[16] Venter, G. (2004), Capital allocation survey with commentary. North American Actuarial Journal, 8, 96-107.

[17] Walhin, J.F. (2002), Some comments on the pricing of exotic excess of loss treaties. Journal of Actuarial Practice, 10, 175-191.

[18] Witdouck, S. and Walhin, J.F. (2004), Some Remarks on the Cedent's Retention Risk in Presence of an Annual Aggregate Deductible or Reinstatements. Blatter Deutsche Gesellschaft fur Versicherungsmathematik, 26-3, 461-481.

[19] Walhin, J.F. and Denuit, M. (2005), On the practical pricing of Top & Drop excess of loss covers. Journal of Actuarial Practice, 12, 137-156.

Appendix A : Results from Matrix Theory

Definition 1. A $n \times n$ symmetric matrix $\boldsymbol{M}$ is positive semidefinite if

$$\boldsymbol{x}^t \cdot \boldsymbol{M} \cdot \boldsymbol{x} \geq 0$$

for all nonzero $n \times 1$ vectors $\boldsymbol{x}$.

Theorem 2. *A $n \times n$ symmetric matrix $\boldsymbol{M}$ is positive semidefinite if and only if all its eigenvalues are nonnegative.*

Definition 3. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two $n \times n$ matrices. If there exists a nonsingular matrix $\boldsymbol{S}$ such that

$$\boldsymbol{B} = \boldsymbol{S} \cdot \boldsymbol{A} \cdot \boldsymbol{S}^t,$$

then $\boldsymbol{B}$ is said to be congruent to $\boldsymbol{A}$.

Definition 4. Let $\boldsymbol{A}$ be a symmetric $n \times n$ matrix. The inertia of $\boldsymbol{A}$ is the ordered triple

$$i(\boldsymbol{A}) = (i_+(\boldsymbol{A}), i_-(\boldsymbol{A}), i_0(\boldsymbol{A}))$$

where $i_+(\boldsymbol{A})$ is the number of positive eigenvalues of $\boldsymbol{A}$, $i_-(\boldsymbol{A})$ is the number of negative eigenvalues of $\boldsymbol{A}$ and $i_0(\boldsymbol{A})$ is the number of zero eigenvalues of $\boldsymbol{A}$.

Theorem 5. *Sylvester's law of inertia : Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two symmetric $n \times n$ matrices. There is a non-singular $n \times n$ matrix $\boldsymbol{S}$ such that $\boldsymbol{A} = \boldsymbol{S} \cdot \boldsymbol{B} \cdot \boldsymbol{S}^t$ if and only if $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same inertia, that is, the same number of positive, negative and zero eigenvalues.*

Definition 6. A $n \times n$ matrix $\boldsymbol{P}$ is called a permutation matrix if exactly one entry in each row and column is equal to 1, and all other entries are 0.
Multiplication by such matrices effects a permutation of the rows or columns of the object multiplied. Left multiplication of a matrix $\boldsymbol{A}$ by a permutation matrix $\boldsymbol{P}$ permutes the rows of $\boldsymbol{A}$ while right multiplication of a matrix $\boldsymbol{A}$ by a permutation matrix $\boldsymbol{P}$ permutes the columns of $\boldsymbol{A}$.

Theorem 7. *The determinant of a permutation matrix is either 1 or -1 so that permutation matrices are nonsingular.*

Theorem 8. *If $\boldsymbol{P}$ is a $n \times n$ permutation matrix, then*

$$\boldsymbol{P}^{-1} = \boldsymbol{P}^t.$$

Theorem 9. *Let $\boldsymbol{P}$ be a $n \times n$ permutation matrix. Since $\boldsymbol{P}^{-1} = \boldsymbol{P}^t$ permutes columns in the same way that the permutation matrix $\boldsymbol{P}$ permutes rows, the transformation $\boldsymbol{A} \to \boldsymbol{P} \cdot \boldsymbol{A} \cdot \boldsymbol{P}^t$ permutes the rows and columns of $\boldsymbol{A}$ in the same way.*

Definition 10. The Householder transformation is a $n \times n$ matrix $\boldsymbol{H}$ given by

$$\boldsymbol{H} = \boldsymbol{I}_n - 2\frac{\boldsymbol{v} \cdot \boldsymbol{v}^t}{\boldsymbol{v}^t \cdot \boldsymbol{v}}$$

where $\boldsymbol{I}_n$ is the $n \times n$ identity matrix and $\boldsymbol{e_1} = (1, \underbrace{0, \ldots, 0}_{n-1 \text{ times}})^t$.

The Householder transformation is such that

$$\boldsymbol{H} \cdot \boldsymbol{x} = \|x\|_2 \boldsymbol{e_1}$$

where $\|x\|_2 = \sqrt{\boldsymbol{x}^t \cdot \boldsymbol{x}}$.
Indeed we have

$$\begin{aligned}
\boldsymbol{H} &= \boldsymbol{I}_n - 2\frac{\boldsymbol{v} \cdot \boldsymbol{v}^t}{\boldsymbol{v}^t \cdot \boldsymbol{v}} \\
\boldsymbol{H} \cdot \boldsymbol{x} &= \boldsymbol{x} - 2\frac{\boldsymbol{v}^t \cdot \boldsymbol{x}}{\boldsymbol{v}^t \cdot \boldsymbol{v}}\boldsymbol{v}
\end{aligned}$$

If $\boldsymbol{v} = \boldsymbol{x} + \alpha \boldsymbol{e_1}$ then

$$\begin{aligned}
\boldsymbol{v}^t \cdot \boldsymbol{x} &= \boldsymbol{x}^t \cdot \boldsymbol{x} + \alpha x_1 \\
\boldsymbol{v}^t \cdot \boldsymbol{v} &= \boldsymbol{x}^t \cdot \boldsymbol{x} + 2\alpha x_1 + \alpha^2 \\
\boldsymbol{H} \cdot \boldsymbol{x} &= \left(1 - 2\frac{\boldsymbol{x}^t \cdot \boldsymbol{x} + \alpha x_1}{\boldsymbol{x}^t \cdot \boldsymbol{x} + 2\alpha x_1 + \alpha^2}\right) \cdot \boldsymbol{x} - 2\alpha \frac{\boldsymbol{v}^t \cdot \boldsymbol{x}}{\boldsymbol{v}^t \cdot \boldsymbol{v}} \boldsymbol{e_1}
\end{aligned}$$

where x_1 is the first entry of the vector $\boldsymbol{x}$.
Assume that $\alpha = -\|x\|_2$ then $\boldsymbol{H} \cdot \boldsymbol{x} = \|x\|_2 \boldsymbol{e_1}$.
In our case $\boldsymbol{x} = \boldsymbol{1}_n$ which implies $\|x\|_2 = \sqrt{n}$.
A Householder transformation is orthogonal :

$$\boldsymbol{H}^t \cdot \boldsymbol{H} = \boldsymbol{I}_n = \boldsymbol{H} \cdot \boldsymbol{H}^t,$$

implying that the Householder transformation is nonsingular.

Theorem 11. *Let $\boldsymbol{A}_i$ denote the $i \times i$ principal submatrix of $\boldsymbol{A}$ determined by the first i rows and columns of $\boldsymbol{A}$. A $n \times n$ symmetric matrix $\boldsymbol{A}$ is positive semidefinite matrix if and only if $det(\boldsymbol{A}_i) \geq 0$ for $i = 1, 2, \ldots, n$.*

Appendix B : Detailed Calculations for the Analysis of Positive Semidefiniteness of $\boldsymbol{Z}$

$$
\begin{aligned}
det(\boldsymbol{Z}_3) \quad = \quad & (1-\rho_{11})(1-\rho_{22})(1-\rho_{33})\frac{1}{n_1 n_2 n_3} \\
& +\rho_{33}(1-\rho_{11})(1-\rho_{22})\frac{1}{n_1 n_2} \\
& +\rho_{22}(1-\rho_{11})(1-\rho_{33})\frac{1}{n_1 n_3} \\
& +\rho_{11}(1-\rho_{22})(1-\rho_{33})\frac{1}{n_2 n_3} \\
& +(1-\rho_{11})(\rho_{22}\rho_{33}-\rho_{23}^2)\frac{1}{n_1} \\
& +(1-\rho_{22})(\rho_{11}\rho_{33}-\rho_{13}^2)\frac{1}{n_2} \\
& +(1-\rho_{33})(\rho_{11}\rho_{22}-\rho_{12}^2)\frac{1}{n_3} \\
& +2\rho_{12}\rho_{13}\rho_{23}+\rho_{11}\rho_{22}\rho_{33}-\rho_{11}\rho_{23}^2-\rho_{22}\rho_{13}^2-\rho_{33}\rho_{12}^2.
\end{aligned}
$$

$$
\begin{aligned}
det(\boldsymbol{Z}_2) \quad = \quad & (1-\rho_{11})(1-\rho_{22})\frac{1}{n_1 n_2} \\
& +\rho_{11}(1-\rho_{22})\frac{1}{n_1} \\
& +\rho_{22}(1-\rho_{11})\frac{1}{n_2} \\
& \rho_{11}\rho_{22}-\rho_{12}^2.
\end{aligned}
$$

$$
\begin{aligned}
det(\boldsymbol{Z}_1) \quad = \quad & (1-\rho_{11})\frac{1}{n_1} \\
& +\rho_{11}.
\end{aligned}
$$