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Title: Premium Determination Based On Change of Measure

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Abstract: This paper discusses a premium determination principle based on the notion of a change of measure. By selecting the mean as premium based on a transformed measure, one can adhere to expectation principle with linearity property. The notion of change of measure is discussed and illustrated by a number of examples based on the application of Radon-Nikodym Theorem. Using Likelihood Ratios, as Radon-Nikodym derivatives, one can provide a unified approach to change of measure problems as illustrated by current research activities in this area.

Key words: Change of Measure, Radon-Nikodym Theorem, expectation principle, distorted probability distributions, Likelihood Ratios, Esscher principle, exponential family, premium calculation principles.

Premium Determination Based On Change of Measure

Summary

This paper discusses a premium determination principle based on the notion of a change of measure. By selecting the mean as premium based on a transformed measure, one can adhere to expectation principle with linearity property. The notion of change of measure is discussed and illustrated by a number of examples based on the application of Radon-Nikodym Theorem. Using Likelihood Ratios, as Radon-Nikodym derivatives, one can provide a unified approach to change of measure problems as illustrated by current research activities in this area.

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1. Introduction: Change of Measure For Denumerable Sample Spaces

Actuaries, economists and financial engineers are involved in pricing insurance and financial risks. They rely on available theories and guiding principles to achieve their tasks. In particular, they use probability theory as the vehicle for dealing with uncertainty. Furthermore, it is recognized that prices of insurance and financial risks are not solely determined by mathematics or economic theories but are also impacted by activities taking place in financial markets. There are similarities as well as differences between pricing insurance and financial risks. The application of probability theory is prevalent in the field of insurance, economics and finance. There are differences in terms of terminology used as well as emphasis on items treated on the liability side as compared to asset side of a balance sheet. Economists and financial engineers rely upon such theories as general equilibrium pricing for products and services in competitive markets, CAPM, Arbitrage Theory, and Option theory as applicable to derivative securities. Actuaries apply ruin theory, and a number of “premium determination principles” in computing premiums for insurance risks. There is considerable interest among these researchers to apply the same framework, a unified approach, to price financial as well as insurance risks. This paper outlines an approach for premium determination based upon the notion of change of measure. The notion of change of measure has also been applied in pricing financial risks. The concept of change of measure is based on a fundamental theorem from measure theory known as Radon-Nikodym theorem. A rigorous treatment of probability relies on use of measure theory. We discuss Radon-Nikodym in this section as well as next depending upon whether we are relying on simple mathematics, or applying tools from measure theory. Each has its merits.

In this section, in order to keep the mathematics manageable, I shall confine myself to an example when the sample space Ω is finite with three outcomes. The ideas in this section can be extended to the case when Ω is infinitely countable. The application of change of measure, Radon-Nikodym Theorem, for the case of finite sample space avoids many intricacies of measure theory. This example provides **transparency**,

and furthermore gives an **explicit** expression for the Radon-Nikodym derivative. Section 2, states the Radon-Nikodym theorem for the general case of non-denumerable sample spaces.

Let Ω be finite sample space, specifically $\Omega = \{\omega_1, \omega_2, \omega_3\}$. A probability measure, P , is a non-negative set function defined on \mathcal{F} , a set of subsets of Ω . \mathcal{F} is a σ -algebra. When Ω is finite, then $\mathcal{F} = \mathcal{P}(\Omega)$, the power set of Ω --the set of all subsets of Ω . In this case, we can define probabilities for all subsets of Ω . In our example, the events that belong to \mathcal{F} are:

$\Omega = \{\omega_1, \omega_2, \omega_3\}$, sample space (sure event),

ϕ , null event (empty set)

$A_1 = \{\omega_1\}$, $A_2 = \{\omega_2\}$, $A_3 = \{\omega_3\}$, are singletons, sets with only one outcome,

$B_1 = \{\omega_1, \omega_2\}$, $B_2 = \{\omega_1, \omega_3\}$, $B_3 = \{\omega_2, \omega_3\}$.

Note the cardinality of \mathcal{F} is $8 = 2^3$, and we write $\mathcal{F} = \{\phi, A_1, A_2, A_3, B_1, B_2, B_3, \Omega\}$.

In order to speak of change of measure, we need, at least, to consider two probability measures defined on the same measure space (Ω, \mathcal{F}) . Let us denote these probability

measures by P and Q respectively. Thus, we have two probability spaces

(Ω, \mathcal{F}, P) and (Ω, \mathcal{F}, Q) .

Next we want to spell out what we “**mean**” by changing from one measure to another.

For (Ω, \mathcal{F}) , we define P , our first probability measure, on \mathcal{F} as follows:

$p_1 = P(A_1)$, $p_2 = P(A_2)$, $p_3 = P(A_3)$ where $0 < p_i < 1$, for $i = 1, 2, 3$, and $\sum_i p_i = 1$

Next, we introduce a second probability measure Q on (Ω, \mathcal{F}) as follows:

$q_1 = Q(A_1)$, $q_2 = Q(A_2)$, $q_3 = Q(A_3)$ where $0 < q_i < 1$, for $i = 1, 2, 3$, and $\sum_i q_i = 1$

The following table lists events of interest with their associated probability measures under P and Q respectively.

Event	Prob. Of Event Under P	Prob. Of Event Under Q
ϕ	0	0
$A_1 = \{\omega_1\}$	p_1	q_1
$A_2 = \{\omega_2\}$	p_2	q_2
$A_3 = \{\omega_3\}$	p_3	q_3
$B_1 = \{\omega_1, \omega_2\}$	$p_1 + p_2$	$q_1 + q_2$
$B_2 = \{\omega_1, \omega_3\}$	$p_1 + p_3$	$q_1 + q_3$
$B_3 = \{\omega_2, \omega_3\}$	$p_2 + p_3$	$q_2 + q_3$
$\Omega = \{\omega_1, \omega_2, \omega_3\}$	1	1

It is worth noting at the outset that P & Q agree for the events ϕ (null event) and Ω (sure event).

Let us now explore the relationship between P & Q for the other events in \mathcal{F} .

We note that

$$Q(A_1) = q_1 = \frac{q_1}{p_1} p_1 = \frac{q_1}{p_1} P(A_1) \quad (1)$$

Similar expressions can be written for $Q(A_2)$ and $Q(A_3)$.

Next, we have

$$\begin{aligned} Q(B_1) &= q_1 + q_2 = \frac{q_1}{p_1} p_1 + \frac{q_2}{p_2} p_2 \\ &= \left\{ \frac{q_1}{p_1} \frac{p_1}{(p_1 + p_2)} + \frac{q_2}{p_2} \frac{p_2}{(p_1 + p_2)} \right\} P(B_1) \end{aligned} \quad (2)$$

Similar expressions can be written for $Q(B_2)$ and $Q(B_3)$.

We now define a non-negative random variable Z on Ω according to the following rule:

$$Z(\omega_1) = \frac{q_1}{p_1}, Z(\omega_2) = \frac{q_2}{p_2}, Z(\omega_3) = \frac{q_3}{p_3}.$$

Then, Z can be written as

$$Z(\omega) = \frac{q_1}{p_1} I_{A_1}(\omega) + \frac{q_2}{p_2} I_{A_2}(\omega) + \frac{q_3}{p_3} I_{A_3}(\omega) \quad (3)$$

$$\text{where } I_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{if } \omega \in A^c \end{cases}$$

I_A , is an indicator variable (a binary variable) and A^c denotes the complement of event A .

With Z as defined by (3) above, we shall re-write (1) as follows

$$Q(A_1) = E_P(Z I_{A_1}) = E_P(Z | A_1) P(A_1) \quad (1.a)$$

Where E_P is the expectation operator based on measure P , and $E_P(Z | A_1)$ is the conditional expectation of Z given the event A_1 has occurred. Note, given that A_1 has occurred, then Z can only take one value.

Similarly, we can re-write (2) as

$$Q(B_1) = E_P(Z I_{B_1}) = E_P(Z | B_1) P(B_1) \quad (2.a)$$

Note that given B_1 has occurred, then Z can potentially be either $\frac{q_1}{p_1}$ or $\frac{q_2}{p_2}$, and we need to **re-weight** the probabilities assigned to ω_1 and ω_2 .

We can now write

$$Q(C) = E_P(Z I_C) = E_P(Z | C) P(C) \quad (4)$$

where $C \in \{A_1, A_2, A_3, B_1, B_2, B_3\}$.

We note that (4) is also valid for the event $C = \Omega$ (sure event) since we have

$$E_P(Z) = 1 \quad (5)$$

Finally, by noting that P & Q agree on ϕ (the null event with probability 0), so we can extend (4) to

$$Q(C) = E_P(Z I_C) = E_P(Z | C) P(C) \quad (6)$$

for $C \in \mathcal{F} = \{\phi, A_1, A_2, A_3, B_1, B_2, B_3, \Omega\}$.

In (6), it is our understanding that $E_P(Z | \phi)$ may be arbitrary defined. But, in this instance, both the left hand side and right hand side of equation (6) are both zero.

So we re-write (6) as

$$Q(C) = E_Q(I_C) = E_P(Z I_C) \quad (7)$$

for all events $C \in \mathcal{F}$.

Let us, now, define a “genuine” random variable Y on Ω as follows:

$$Y(\omega_1) = y_1, Y(\omega_2) = y_2, Y(\omega_3) = y_3$$

Although Z and Y are both random variables defined on Ω , we would like to view Y as a “genuine” (realistic) random variable on Ω while Z is a “mathematical construct” used to serve a useful function in explaining a change of measure from P to Q .

We note that Y has two alternative probability distributions on Ω depending on measure P or Q respectively. We can now give an expression for the expectation of Y under measure Q in terms of P and Z as follows:

$$\begin{aligned} E_Q(Y) &= \sum_{\omega_i} Y(\omega_i) Q(\omega_i) = \sum_{\omega_i} y_i q_i \\ &= \sum_{\omega_i} y_i \frac{q_i}{p_i} p_i = E_P(Z Y) \end{aligned}$$

Implying that

$$E_Q(Y) = E_P(ZY) \quad (8)$$

Let us summarize, our results in this section as follows:

- a. We considered a finite sample space $\Omega = \{\omega_1, \omega_2, \omega_3\}$
- b. Two probability measures on (Ω, \mathcal{F}) were defined, given rise to two probability spaces (Ω, \mathcal{F}, P) and (Ω, \mathcal{F}, Q) respectively.
- c. We noted that P and Q agreed on ϕ the null event with probability zero. This point becomes much more relevant in the Section 2.
- d. We assumed that probabilities assigned to $A_1, A_2, \text{ and } A_3$ are strictly positive under P & Q . This requirement avoids dividing by zero.
- e. We created a non-negative random variable Z , a “mathematical construct”, such that we can compute probabilities or expectations, see equations (7) and (8), based on Q in terms of P and Z . Furthermore, for Z , we have $E_P(Z) = 1$.
- f. In anticipation of Radon-Nikodym theorem, Z as defined above is called a Radon-Nikodym derivative. It is symbolically written as $Z = \frac{dQ}{dP}$. In our case it is more appropriate to think of Z as $\frac{Q(A)}{P(A)}$, the quotient of two probability measures.
- g. Furthermore, we have $E_Q(Y) = E_P(ZY)$. Note that when Ω is finite, there is no issue of integrability of Y .

We can now proceed to discuss the notion of change of measure when Ω is not necessarily denumerable.

2. Change of Measure: The Radon-Nikodym Theorem

Change of measure is a fundamental theorem in measure theory known as Radon-Nikodym Theorem. We shall state a version of Radon-Nikodym Theorem suitable for probability measures as referenced by Jacod and Protter (2002), Theorem 28.3, page 246, with proof provided in Chapter 14 of Williams (1991).

Radon-Nikodym Theorem:

Let P be a probability on measure space (Ω, \mathcal{F}) and let Q be a finite measure on (Ω, \mathcal{F}) . If $Q \ll P$ then there exists a nonnegative random variable Z such that

$$Q(C) = E_P(Z I_C) \text{ for all } C \in \mathcal{F}. \text{ Moreover } Z \text{ is } P\text{-unique a.s. We write } Z = \frac{dQ}{dP},$$

Z is referred to as Radon-Nikodym derivative.

We now make the following comments regarding the Radon-Nikodym Theorem:

- a. Since Q is a finite measure, meaning that $Q(\Omega)$ is finite, then we can normalize Q , by $\frac{Q(A)}{Q(\Omega)}$, so that Q will be a probability measure (i.e., $Q(\Omega)=1$). The normalization implies that $E_p(Z) = 1$.
- b. $Q \ll P$ means that the measure Q is absolutely continuous with respect to the measure P . That is $Q(A) = 0$ whenever $P(A) = 0$. Thus, P and Q agree on events that have zero probability. These are sets of measure zero.
- c. P -unique a.s. means that if there is another version of Z , say Z^* , then we have $P(\{\omega : Z(\omega) = Z^*(\omega)\}) = 1$. That is the set when Z and Z^* don't agree has a measure zero.

- d. If we start with a probability measure P and are given a nonnegative measurable function, Z , then it is not hard to show that $Q(A)$ as define by

$$Q(A) = \int_A Z(\omega) dP(\omega)$$

is a set function and also a measure. The

importance or significance of Radon-Nikodym theorem is the given two measures Q and P , with $Q \ll P$, then there **exists** a measurable function Z such that Q and P are related according to $Q(A) = \int_A Z(\omega) dP(\omega)$. Note that

the Radon-Nikodym states the existence of Z without providing an explicit expression for Z . In the case when Ω is denumerable we can construct Z explicitly as illustrated in Section 2.

- e. Furthermore, if we assume that Y is integrable, then we have

$$\int_A Y(\omega) dQ(\omega) = \int_A Y(\omega) Z(\omega) dP(\omega)$$

which we shall write as

$$E_Q(Y) = E_P(ZY)$$

The results based on Radon-Nikodym derivative, $Z = \frac{dQ}{dP}$, are summarized below:

$$a. Q(C) = E_Q(I_C) = E_P(Z I_C) \quad (7)$$

$$b. E_Q(Y) = E_P(ZY), \quad (8)$$

Note that equation (7) is a statement of the Radon-Nikodym theorem, and equation (8) is an extension of Radon-Nikodym theorem to an integrable r.v. Y .

Let us proceed to use the notion of a change of measure as a basis of determining premium for insurance risks.

3. A Premium Determination Principle based on notion of change of measure

Pricing insurance risks is involved. It requires the modeling of frequency, severity, expenses and profit load. The actuary should be cognizant of activities taking place in insurance “markets”. Insurance risks tend to be heterogeneous and the usual assumption of “independent identically distributed” r.v.’s is usually not tenable. Furthermore, historical loss and exposure data need to be adjusted in order to estimate model parameters. For instance, in order to determine a severity—size of loss—distribution, reported losses need to be adjusted for further development and trended to reflect current values. Furthermore, estimation of model parameters requires applying methods suitable for incomplete data, see Guiahi (2001).

Let Y be a r.v. that describes an important characteristics of a risk, for instance the severity. By re-writing (8) in an alternative format, we obtain an interesting result for determining premium for insurance risks.

$$E_Q(Y) = E_P(ZY) = Cov_P(Z, Y) + E_P(Z)E_P(Y) \quad (9)$$

By noting that $E_P(Z) = 1$, see the comments made after the statement of Radon-Nikodym theorem in Section 2, then (9) can be written as

$$E_Q(Y) = E_P(Y) + Cov_P(Z, Y) \quad (10)$$

Equation (10) has many implications for premium determination.

If Y and Z are positively correlated random variables under probability measure P , then (10) implies

$$E_Q(Y) \geq E_P(Y) \quad (11)$$

For insurance risks, normally, premium exceeds pure premium, $E_P(Y)$.

If we define premium by

$$\text{Premium}(Y) = E_Q(Y) = E_P(Y) + Cov_P(Z, Y) \quad (12)$$

Then, premium is based on an expectation principle using an adjusted probability measure Q . Venter (1991) had suggested the use of an adjusted distribution as a basis of premium calculation. Also, Wang (1996), and Wang (2000) provide for specific adjusted probability distributions to compute premium.

If we use equation (12), as basis of premium determination, then we may note that pure premium $E_P(Y)$ is based on a risk’s attribute through Y (P measure), while the “premium” depends on Q measure which is a function of two variables P (P-measure) and Z (Radon-Nikodym derivative). It is also interesting to note that the return for a stock, based on CAPM, uses the stock risk performance as well as Market return. Bühlmann (1980), “economic principle” also suggests the use of auxiliary information in addition to risk information to calculate premium.

From equation (12), we can define risk load as

$$\text{Risk Load} = \text{Cov}_P(Z, Y) \quad (13)$$

Re-visiting (12), we see that we have combined the two principles of “Adjusted distribution principles” and “Covariance principles” as referred to by Venter (1991) in a single equation given by (12).

An advantage of applying (12) as the basis of premium determination is as follows:

Suppose $Y = Y_1 + Y_2$, then we re-write (12) as

$$E_Q(Y) = E_P(Y_1 + Y_2) + \text{Cov}_P(Z, Y_1 + Y_2) \quad (14)$$

But

$$\text{Cov}_P(Z, Y_1 + Y_2) = \text{Cov}_P(Z, Y_1) + \text{Cov}_P(Z, Y_2) \quad (15)$$

Combining (14) and (15) and noting the expectation operator is additive for random variables, we get the following result.

$$\begin{aligned} \text{Premium}(Y_1 + Y_2) &= E_Q(Y_1 + Y_2) \\ &= \{E_P(Y_1) + \text{Cov}_P(Z, Y_1)\} + \{E_P(Y_2) + \text{Cov}_P(Z, Y_2)\} \\ &= \text{Premium}(Y_1) + \text{Premium}(Y_2) \end{aligned} \quad (16)$$

It is interesting to note that (16) is valid irrespective of whether Y_1 and Y_2 are dependent or independent under measure P .

Premium principles based on second moments (variance or standard deviation risk loads), will not adhere to additive property of premiums for **layered** risk, see Venter (1991). We can apply (16) to layered risk by noting that

$$Y = Y I_{\{Y \leq D\}} + Y I_{\{Y > D\}} = Y_1 + Y_2$$

where D is a deductible or retention, and Y_1 represents a primary layer loss and Y_2 represents an excess layer loss. It should be noted that the expectation principle is based on Q-measure and not P -measure. It may be difficult to explain to an insured client why the premium for two non-overlapping layer of insurance do not add up!

Also, based on the assumption of the absence of arbitrage, we would like that price for risk Y_1 and price for risk Y_2 should be the same as price for the portfolio $Y_1 + Y_2$. So, the expectation principle based on Q-measure is consistent with additivity of price for risks. It is worth mentioning that no arbitrage rule applies when there are no frictional cost in the market. When there is friction in the market, as it is for the insurance

market, then the addition rule may be replaced by a premium rule that allows sub-additivity, see Chateauneuf, Kast, and Lapied (1996) for some interesting examples.

Two perspectives regarding Radon-Nikodym theorem. First, if we are given two measures P and Q , then we know that these measures are related to each other by

Radon-Nikodym derivative, $Z = \frac{dQ}{dP}$, based on relationship $Q(A) = \int_A Z(\omega) dP(\omega)$.

A second perspective on Radon-Nikodym Theorem would be to start with a probability measure P , then consider a “weight” function Z , Z being nonnegative and having an expected value of one under the P -measure. Then, we proceed construct a **new measure** Q by using the relationship $Q(A) = \int_A Z(\omega) dP(\omega)$. This second perspective on Radon-

Nikodym Theorem may be more useful in pricing insurance risks. I shall illustrate this point of view by giving some examples.

Example 1: P -measure as an Exponential Distribution

Suppose risk Y has an Exponential Distribution according to P -measure with parameter α . Y could present the severity distribution of a risk.

We have $f_Y(y) = \alpha e^{-\alpha y}$ with mean $\frac{1}{\alpha}$, where f_Y is the density for Y .

Now, let $Z = Z(y)$ be a “weight” function of the form $k e^{by}$. In order that this weight function be nonnegative, then, k should be positive. Furthermore, we want

$E_P(Z) = 1$ which implies that $k = \frac{1}{M_Y(b)}$ where $M_Y(b)$ is the moment generating

function, m.g.f. for Y evaluated at b .

Then,

$$Z(y)f_Y(y) = \frac{e^{by}}{M_Y(b)} f_Y(y) = \frac{e^{by}}{M_Y(b)} \alpha e^{-\alpha y} \quad (17)$$

Since the m.g.f. of Exponential distribution is $M_Y(b) = \frac{\alpha}{\alpha - b}$, we can re-write the right-hand-side of equation (17) as

$$(\alpha - b)e^{-(\alpha - b)y} \quad (18)$$

Thus Q -measure for Y , relationship (18), is an Exponential with parameter $\alpha - b$ instead of α . It is interesting to note that Q has the same type of distribution as P .

Let us use these results to illustrate a pricing example. Suppose that we require a profit load of 25%, say. Then, we can write for the Risk Load = $1 + c$, where $c = .25$ (or 25%).

From (12), if we have $1 + c = \frac{E_Q(Y)}{E_P(Y)} = \left(\frac{1}{\alpha - b}\right) / \left(\frac{1}{\alpha}\right) = \frac{\alpha}{\alpha - b}$, then for a Risk Load

of $c=25\%$, we can choose $b = \frac{c}{1 + c} \alpha$

We obtained a new measure Q , a new distribution for Y , by weighing the original distribution, P -measure, by an exponential function.

Note that

$$e^{-(\alpha-b)c} = Q(Y > c) > P(Y > c) = e^{-\alpha c}$$

Thus, Y is stochastically “larger” under the new measure (distribution)—heavier tail!

Example 2: P -measure as a Normal Distribution

Let us give an example of change of measure from the field of finance. An asset’s return is commonly modeled in finance by using a normal distribution. Note that if the return of an asset is assumed to be normal, then the asset price would have a lognormal distribution. If we view the return of an asset over a period of time, then return process would be modeled by a Brownian motion, and the stochastic process for asset prices is modeled as a geometric Brownian motion (a diffusion process). For reference to geometric Brownian motion see texts on financial economics, for example Etheridge (2002).

Let us establish a result about normal distribution which we shall apply shortly. Denote by $N(\mu, \sigma^2)$, a normal r.v. with mean μ and variance σ^2 . Let Y represent an attribute of a risk under study which has a normal distribution under P -measure. Let us apply the exponential weight of the previous example, as Radon-Nikodym derivative for Z . Then, we have

$$\frac{e^{by}}{M_Y(b)} f_Y(y) = \frac{e^{by}}{M_Y(b)} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(y-\mu)^2} \quad (19)$$

Recall that the m.g.f. of normal r.v. is $M_Y(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$.

We can re-write (19) as

$$\begin{aligned} \frac{e^{by}}{M_Y(b)} f_Y(y) &= \frac{e^{by}}{e^{\mu b + \frac{1}{2}\sigma^2 b^2}} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(y-\mu)^2} \\ &= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}[y-(\mu+b\sigma^2)]^2} = N(\mu + b\sigma^2, \sigma^2) \end{aligned} \quad (20)$$

Equation (20) is obtained by completing squares (some basic algebra). The right-hand-side of (20) shows that Q -measure for Y is another normal given by $N(\mu + b\sigma^2, \sigma^2)$. Thus, we have obtained another normal distribution by means of shifting the mean to the right by an amount of $b\sigma^2$.

In Black-Scholes option model, see Etheridge (2002), the behaviour of a stock over time, S_t , is a stochastic process governed by a stochastic differential equation (a diffusion process)

$$dS_t = \mu S_t dt + \sigma S_t dW_t \quad (21)$$

where μ and σ are constants and $\{W_t, t \geq 0\}$ is a standard Brownian motion. The solution of equation (21) is given by

$$\begin{aligned} S_t &= S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right) \\ &= S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t} N(0,1)\right) \\ &= \exp\left(\log S_0 + \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t} N(0,1)\right) \end{aligned} \quad (22)$$

We note that for a fixed t , equation (22) states that S_t has lognormal distribution with parameters $\log S_0 + \left(\mu - \frac{1}{2}\sigma^2\right)t$ and $\sigma^2 t$.

We also have the return for S_t as

$$E\left(\frac{S_t}{S_0}\right) = e^{\mu t} \quad (23)$$

In a risk-neutral “world” where assets have the same return, we would expect the return given by (23), would parallel that of risk-free bond, B_t , with return given by

$$E\left(\frac{B_t}{B_0}\right) = e^{rt} \quad (24)$$

where r denotes risk-free return for a zero coupon bond.

We can change the distribution S_t from a P -measure to a Q -measure by a suitable choice of b , so that we would have

$$S_t = \exp\left(\log S_0 + \left(r - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t} N(0,1)\right)$$

which is risk-neutral distribution of S_t . This can be accomplished by selecting

$$b = -\frac{1}{\sigma} \frac{(\mu - r)}{\sigma}, \text{ and using the results given by equations (20) and (22).}$$

Example 3: P -measure as a member of one-parameter Exponential Family

Exponential family of distributions play an important role in statistic. The exponential family encompasses many useful discrete as well as continuous distributions in statistics. They are the underlying distribution for “Generalized Linear Modeling”, see McCullagh and Nelder (1989). Let us consider the canonical form of one-parameter exponential family as presented by Bickel and Doksum (2001). The P -measure is

presented by either a density (continuous case) or a probability mass function (discrete case) as follows

$$f(y; \theta) = h(y) \exp\{\theta T(y) - A(\theta)\}$$

where $T(Y)$ is a function of Y which represents a sufficient statistic (estimator) for parameter θ .

Since the density integrates to one or probability mass function sums to one, we have either

$$e^{A(\theta)} = \int h(y) e^{\theta T(y)} dy \text{ or } \sum_y h(y) e^{\theta T(y)}$$

We also have

$$E[T(Y)] = A'(\theta)$$

Let us concentrate on the continuous case, the discrete case can be treated similarly.

The m.g.f. of $T(Y)$ is given by

$$\begin{aligned} M_{T(Y)}(s) &= \int e^{sT(y)} h(y) e^{\theta T(y) - A(\theta)} dy \\ &= \int h(y) e^{(\theta+s)T(y) - A(\theta+s)} dy e^{A(\theta+s) - A(\theta)} \\ &= e^{A(\theta+s) - A(\theta)} \end{aligned}$$

Let us select Z , the Radon-Nikodym derivative or the exponential weight function as in the previous examples.

Then, we have

$$\begin{aligned} \frac{e^{bT(y)}}{M_{T(Y)}(b)} f_{T(Y)}(y; \theta) &= \frac{e^{bT(y)}}{e^{A(\theta+b) - A(\theta)}} h(y) e^{\theta T(y) - A(\theta)} \\ &= h(y) e^{(\theta+b)T(y) - A(\theta+b)} \end{aligned} \quad (24)$$

Thus (24), Q -measure for $T(Y)$ is also another one parameter exponential with parameter θ replaced by $\theta + b$, i.e., a shift of an amount b .

If we calculate the premium as the mean of the Q -measure, then the

$$\text{Premium} = A'(\theta + b)$$

When the Radon-Nikodym derivative Z is the exponential weight function, then we obtain the same result as that of the Esscher transformation, see Kamps (1998).

Our next example relate the transformation process for a r.v. and its associated Radon-Nikodym derivative.

Example 4: Transformation & Associated Radon-Nikodym Derivative.

Let X be a r.v. of interest with density function f_X . Here, the P -measure is given by f_X . Consider a transformation of X to a r.v. Y according to

$$Y = g(X) \quad (25)$$

Let us assume that $g(x)$ is strictly increasing function of x and differentiable for all x . Furthermore, let us assume that the density of X , f_X , is not zero for any x value.

We have

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(g(X) \leq y) \\ &= P(X \leq g^{-1}(y)), \quad g^{-1} \text{ exists since } g \text{ is a strictly increasing function.} \\ &= F_X(g^{-1}(y)) \end{aligned} \quad (26)$$

Differentiating both sides of equation (26) with respect to y , we get

$$f_Y(y) = f_X(g^{-1}(y)) \frac{1}{g'(g^{-1}(y))} \quad (27)$$

where f_Y is the density of $Y = g(X)$.

Define r.v. Z , based on (27), according to

$$Z = \frac{f_Y(g(X))g'(X)}{f_X(X)} \quad (28)$$

Then, we have

$$\int_A f_Y(y)dy = \int_A Z f_X(x)dx \quad (29)$$

Note that we have $Z \geq 0$ based on (28), and $E_X[Z] = 1$ because of (27). Thus, Z , is a Radon-Nikodym derivative transforming X into Y .

4. A Change of Measure Based on Likelihood Ratio Principle

We begin our description of P -measure by considering a r.v. X with a density f_X used to compute probabilities according to

$$P(A) = \int_A f_X(x)dx. \quad (30)$$

Further, assume that the density of X is strictly positive.

We can create a new measure Q from P by selecting a “weight” function w using

$$Q(A) = \int_A w(x) f_X(x) dx \quad (31)$$

provided that w is nonnegative and $E_P[w(X)] = 1$

The weight w can be considered to be a Radon-Nikodym derivative relating P to Q . This is the second perspective of Radon-Nikodym derivative as referred to in Section 3.

Note that by selecting different w weight functions we can construct different Q -measures based on a specified P -measure.

In Examples 1 and 3 above, we constructed Q from P by using the weight function

$$w(x) = \frac{e^{bx}}{M_X(b)}, \text{ the Escher transformation, where } M_X \text{ is the m.g.f. for } X.$$

Now, we shall describe an alternative way of creating weight functions based on the notion of Likelihood Ratio.

Define a function $H : R \rightarrow R$, where R is the real line with the following properties:

- a. H is strictly increasing
- b. H is differentiable with derivative denoted by h
- c. $H(x) \uparrow 1$ as $x \uparrow \infty$, and $H(x) \downarrow 0$ as $x \downarrow -\infty$

Such H function can present the cumulative distribution function of a given r.v.

We have

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} h(x) dx \\ &= \int_{-\infty}^{\infty} \frac{h(x)}{f_X(x)} f_X(x) dx \end{aligned} \quad (32)$$

We can now define a “large” class of weight functions by

$$w(x) = \frac{h(x)}{f_X(x)} \quad (33)$$

Note that w is nonnegative being the ratio of two densities. Furthermore, $E_P[w(X)] = 1$ because of (32). Hence, w is a version of Radon-Nikodym derivative.

Now we shall give some examples of the function H with its corresponding density h and weight function w as a means of generating alternative new Q -measures for a specified P -measure.

For our first example, we have

$$\begin{aligned} H(x) &= 1 - [1 - F_X(x)]^r, \text{ with } 0 < r \leq 1 \\ h(x) &= r(1 - F_X(x))^{r-1} f_X(x) \end{aligned}$$

$$w(x) = \frac{h(x)}{f_x(x)} = r(1 - F_x(x))^{r-1} \quad (34)$$

This form of $H(x) = 1 - [1 - F_x(x)]^r$ corresponds to Wang (1996) Proportional Hazards Model (PH-transformation). Note that the premise of PH-transformation is that

$$Q(X > t) = \int_t^{\infty} w(x) f_x(x) dx > P(X > t)$$

Thus, X is stochastically “larger” under Q -measure as compare to P -measure.

Our second example is also based on Wang (2000), here we have:

$$\begin{aligned} H(x) &= \Phi(\Phi^{-1}(F_x(x)) + \lambda), \\ h(x) &= \varphi(\Phi^{-1}(F_x(x)) + \lambda) \frac{f_x(x)}{\varphi(\Phi^{-1}(F_x(x)))}, \\ w(x) &= \frac{\varphi(\Phi^{-1}(F_x(x)) + \lambda)}{\varphi(\Phi^{-1}(F_x(x)))} \end{aligned} \quad (35)$$

where Φ is the cumulative distribution of the standard normal, φ is density of the standard normal. It is interesting to note that (35) presents the ratio of two normal densities where the variable x is measured on a different scale.

Finally, we can even cover the case of Essecher transformation by considering

$$\begin{aligned} H(x) &= \int_{-\infty}^x \frac{e^{by}}{M_x(b)} f_x(y) dy \\ h(x) &= \frac{e^{bx}}{M_x(b)} f_x(x) \\ w(x) &= \frac{h(x)}{f_x(x)} = \frac{e^{bx}}{M_x(b)} \end{aligned}$$

Thus, selecting “weight” functions or Radon-Nikodym derivatives as a Likelihood Ratios we have a procedure that is very versatile for creating alternative Q -measures to be considered as the basis of premium computation.

6. Conclusion

By changing the underlying distribution from a P -measure to a Q -measure, one can calculate premium based on the mean of Q -measure. This procedure is based on applying the expectation principle to a new measure. Expectation principle preserves the

additive property for risks. It is important to note that Q -measure is dependent on two variables: P -measure as well as Z , the Radon-Nikodym derivative. From a computational point of view, it is desirable for Q and P to have the same distributional form. The use of Likelihood Ratio is consonant with Radon-Nikodym derivative, providing a unified approach to change of measure problems. The Esscher principle, and Wang's distorted probability distributions can be derived by proper selection of Likelihood Ratios or Radon-Nikodym derivatives. A challenge to actuaries is in selecting suitable weight functions for the pricing problem at hand, so that the premium derived by computing the mean of the Q -measure is appropriate for a particular situation.

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